POLYNOMIALS

A *polynomial in x is a function that is expressible as a sum of finitely many terms of the form* cx^n,
where c is a constant and n is a nonnegative integer.

A general polynomial can be written in either of the following forms,

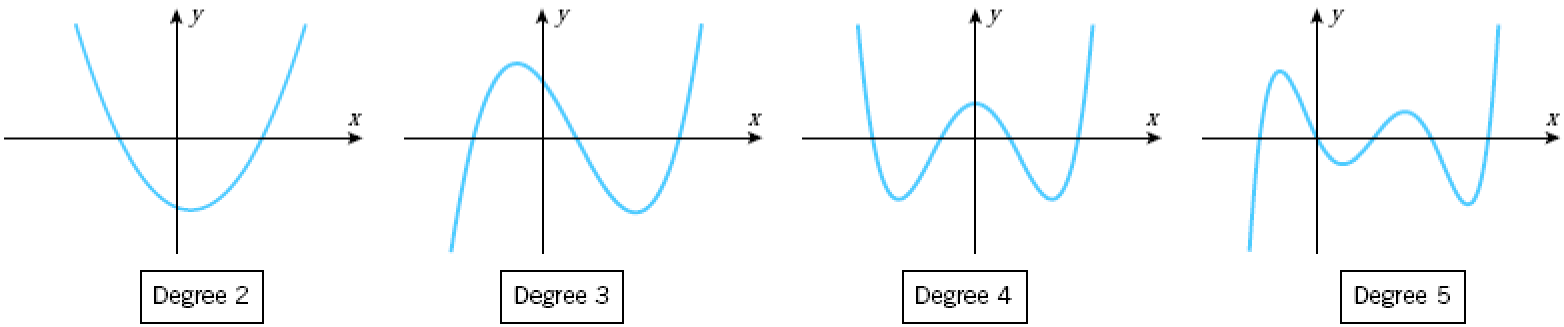
$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

The constants $c_0, c_1, \ldots, c_n$ are called the ***coefficients*** of the polynomial.

The constant 0 is a polynomial called the *zero polynomial*.

Polynomials of degree 1, 2, 3, 4, and 5 are described as *linear, quadratic, cubic, quartic, and quintic, respectively.*



RATIONAL FUNCTIONS

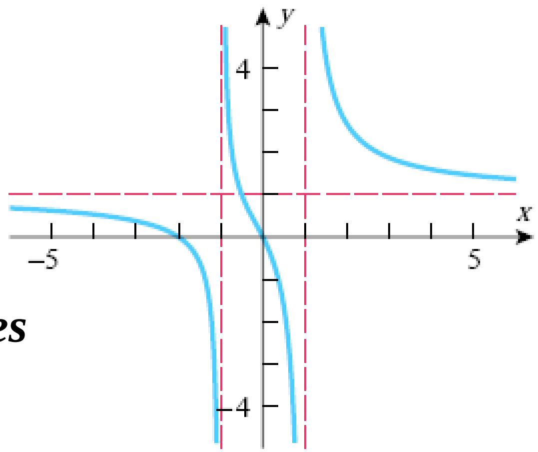
A function that can be expressed as a ratio of two polynomials is called a *rational function.* If *P(x) and Q(x) are polynomials, then the domain of the rational function*

$$f(x) = \frac{P(x)}{Q(x)}$$

For example, the domain of the rational Function

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

consists of all values of *x*, *except x = 1 and x = −1.*



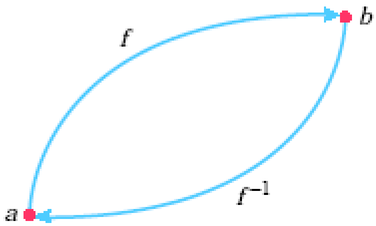
Vertical and horizontal asymptotes

ALGEBRAIC FUNCTIONS

Functions that can be constructed from polynomials by applying finitely many algebraic operations (addition, subtraction, multiplication, division, and root extraction) are called *algebraic functions. Some examples are*

$$f(x) = \sqrt{x^2 - 4}, \quad f(x) = 3\sqrt[3]{x}(2+x), \quad f(x) = x^{2/3}(x+2)^2$$

INVERSE FUNCTIONS



Definition. If the functions *f and g satisfy the two conditions*

g(f(x)) = x for every x in the domain of f

f(g(y)) = y for every y in the domain of g

Then *f is an inverse of g and g is an inverse of f or that f and g are inverse functions. We* can express the equations in Definition

$f^{-1}(f(x)) = x$ for every x in the domain of f

$f(f^{-1}(y)) = y$ for every y in the domain of f^{-1}

domain of f^{-1} = range of f

range of f^{-1} = domain of f

Theorem: ***If an equation y = f(x) can be solved for x as a function of y, say x = g(y), then f has an inverse and that inverse is g(y)=*** $f^{-1}(y)$.

$y = x^3 + 1$ $\boxed{y = f(x)}$

can be solved for *x as a function of y*

$x = \sqrt[3]{y-1}$ $\boxed{x = g(y)}$

Using *x as the* independent variable for both f and f^{-1}

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in the domain of } f^{-1}$$

Example: Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$
$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$
$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x$$

Example: Find a formula for the inverse of $f(x) = \sqrt{3x-2}$ with x as the independent variable, and state the domain of f^{-1}.

Solution

$$y = \sqrt{3x-2}$$

Solve this equation for *x as a function of y*

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2 + 2)$$

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2)$$

$$f^{-1}(x) = \tfrac{1}{3}(x^2 + 2)$$

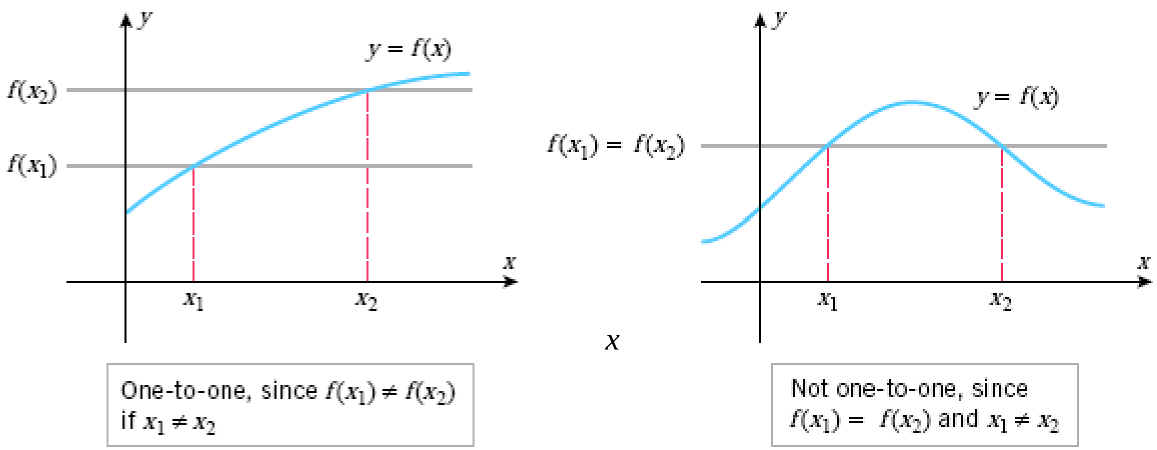
The natural domain of above equation is $(-\infty, +\infty)$**, *whereas the range of f(x) is*** $[0, +\infty)$.

domain of f^{-1} is the range of f.

$$f^{-1}(x) = \tfrac{1}{3}(x^2 + 2), \quad x \geq 0$$

Theorem : *A function has an inverse if and only if it is one-to-one.*

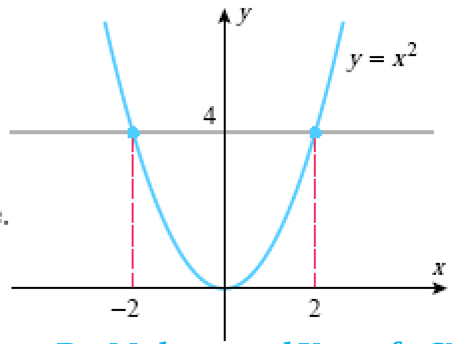
Theorem: (*The Horizontal Line Test) A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*

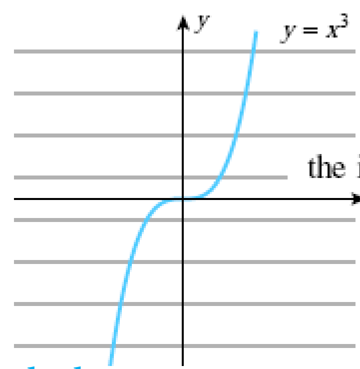


x

Example: **Use the horizontal line test to show that *f(x)* = x^2 *has no inverse but that f(x)* = x^3 *does.***

$f(x) = x^2$ is not invertible.





the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$

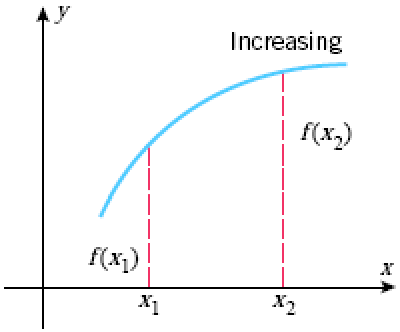
INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

If x_1 and x_2 are points in the domain of a function f, then f is increasing if

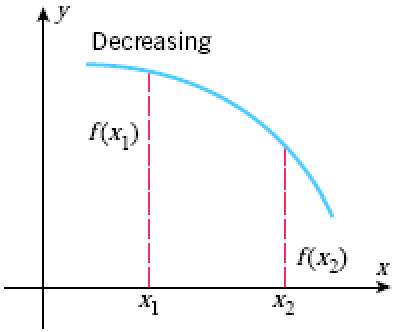
$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2$$

and f is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2$$



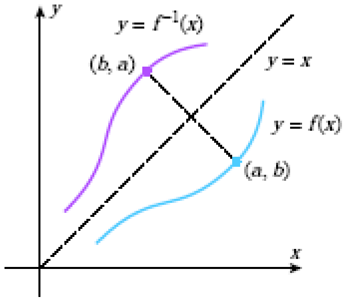
$f(x_1) < f(x_2)$ if $x_1 < x_2$



$f(x_1) > f(x_2)$ if $x_1 < x_2$

GRAPHS OF INVERSE FUNCTIONS

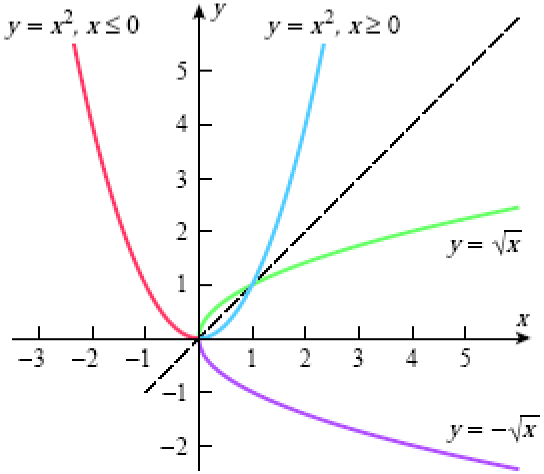
Theorem: *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*



RESTRICTING DOMAINS FOR INVERTIBILITY

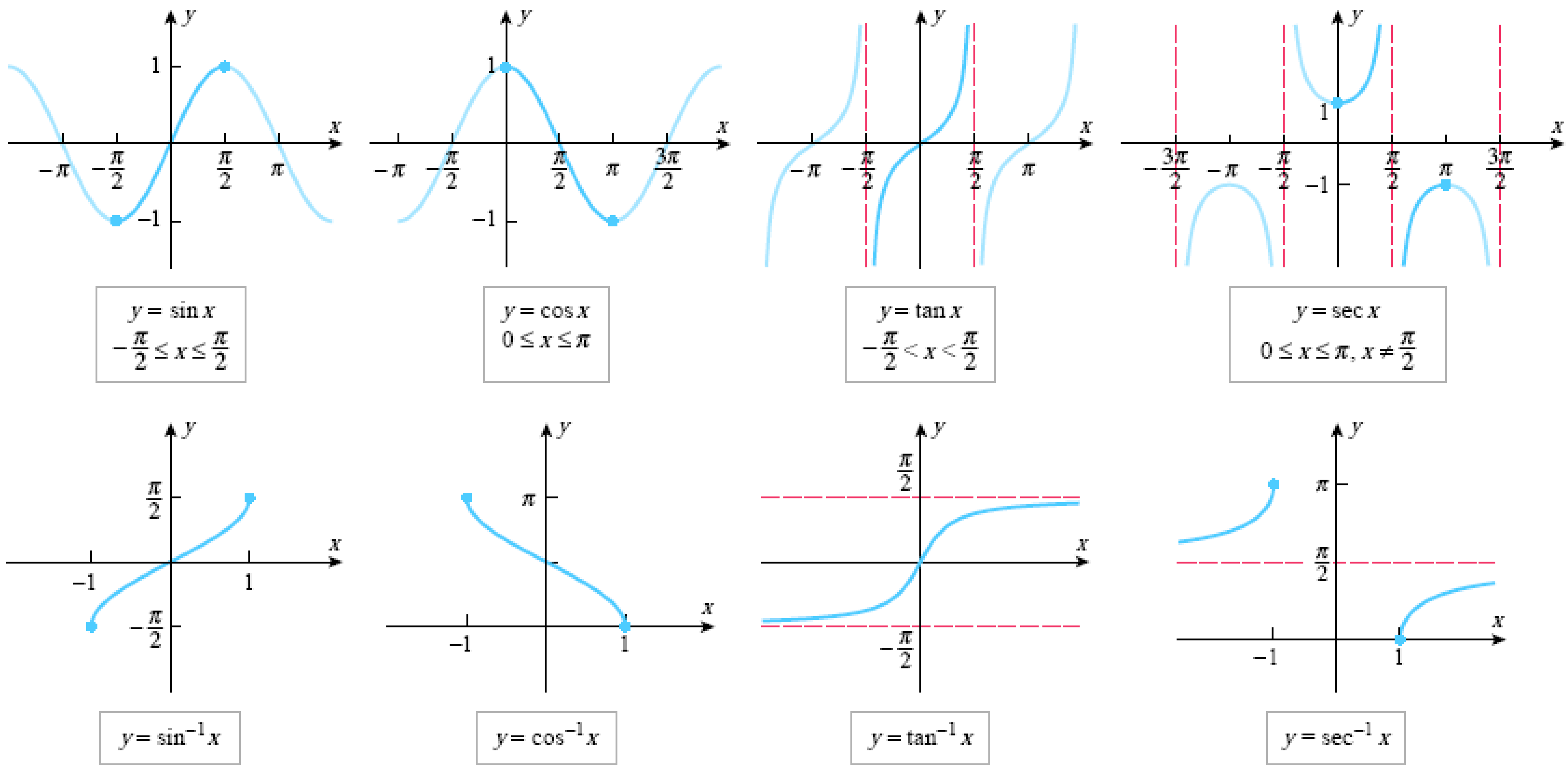
$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

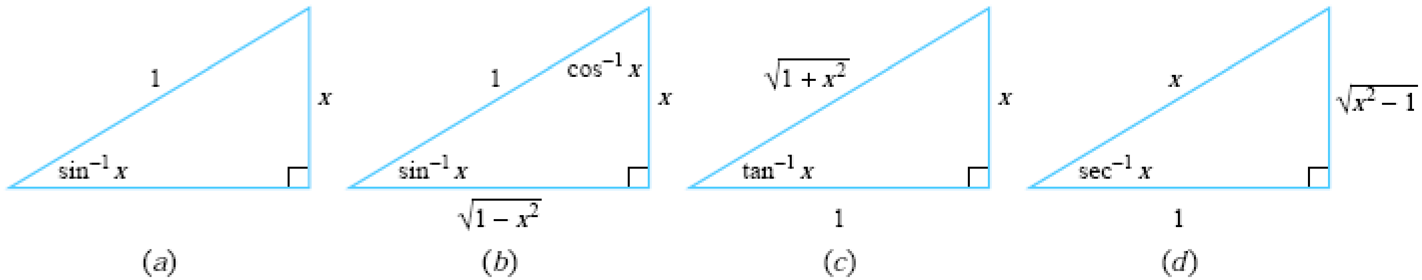
$$f_1^{-1}(x) = \sqrt{x} \quad \text{and} \quad f_2^{-1}(x) = -\sqrt{x}$$



x^2

INVERSE TRIGONOMETRIC FUNCTIONS





FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

identities involving inverse trigonometric functions that are valid for $-1 \leq x \leq 1$;

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$
$$\cos(\sin^{-1} x) = \sqrt{1 - x^2}$$
$$\sin(\cos^{-1} x) = \sqrt{1 - x^2}$$
$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}$$