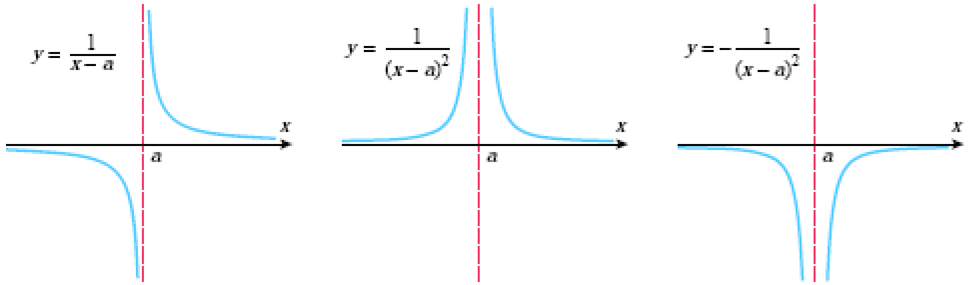
One of the following situations occurs:

- The limit may be $-\infty$ from one side and $+\infty$ from the other.
- The limit may be $+\infty$.
- The limit may be $-\infty$.



$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty$$
$$\lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

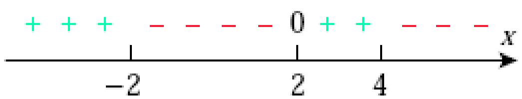
$$\lim_{x \to a} \frac{1}{(x-a)^2} = +\infty$$

$$\lim_{x \to a} -\frac{1}{(x-a)^2} = -\infty$$

Example :

(a) $\displaystyle\lim_{x\to 4^+}\frac{2-x}{(x-4)(x+2)}$ (b) $\displaystyle\lim_{x\to 4^-}\frac{2-x}{(x-4)(x+2)}$ (c) $\displaystyle\lim_{x\to 4}\frac{2-x}{(x-4)(x+2)}$

Solution.



Sign of $\dfrac{2-x}{(x-4)(x+2)}$

(a) $\displaystyle\lim_{x\to 4^+}\frac{2-x}{(x-4)(x+2)}=-\infty$ and (b) $\displaystyle\lim_{x\to 4^-}\frac{2-x}{(x-4)(x+2)}=+\infty$

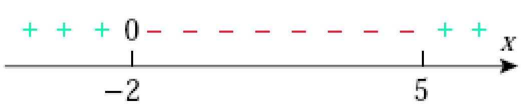
(c) Because the one-sided limits have opposite signs, all we can say about the two-sided limit is that it does not exist.

Example : (a) $\displaystyle\lim_{x\to 3}\frac{x^2-6x+9}{x-3}$ (b) $\displaystyle\lim_{x\to -4}\frac{2x+8}{x^2+x-12}$ (c) $\displaystyle\lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}$

Solution. (a) $\displaystyle\lim_{x\to 3}\frac{x^2-6x+9}{x-3}=\lim_{x\to 3}\frac{(x-3)^2}{x-3}=\lim_{x\to 3}(x-3)=0$

(b) $\displaystyle\lim_{x\to -4}\frac{2x+8}{x^2+x-12}=\lim_{x\to -4}\frac{2(x+4)}{(x+4)(x-3)}=\lim_{x\to -4}\frac{2}{x-3}=-\frac{2}{7}$

(c) $\displaystyle\lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}=\lim_{x\to 5}\frac{(x-5)(x+2)}{(x-5)(x-5)}=\lim_{x\to 5}\frac{x+2}{x-5}$



$\displaystyle\lim_{x\to 5^+}\frac{x^2-3x-10}{x^2-10x+25}=\lim_{x\to 5^+}\frac{x+2}{x-5}=+\infty$ $\displaystyle\lim_{x\to 5^-}\frac{x^2-3x-10}{x^2-10x+25}=\lim_{x\to 5^-}\frac{x+2}{x-5}=-\infty$

Sign of $\dfrac{x+2}{x-5}$

Theorem: ***Let***

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(a) *If* $q(a) \neq 0$, *then* $\lim_{x \to a} f(x) = f(a)$.

(b) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist.*

LIMITS INVOLVING RADICALS

Example Find $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$.

Solution.

$$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1 \quad (x \neq 1)$$

Therefore,

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x \to 1} (\sqrt{x}+1) = 2$$

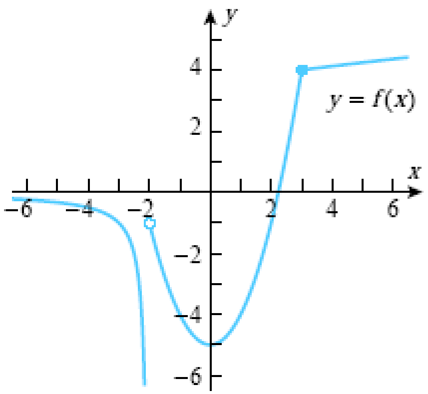
LIMITS OF PIECEWISE-DEFINED FUNCTIONS

Example Let

$$f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \le 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

Find

(a) $\lim_{x \to -2} f(x)$ (b) $\lim_{x \to 0} f(x)$ (c) $\lim_{x \to 3} f(x)$



Solution. (a)

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} = -\infty$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 5) = (-2)^2 - 5 = -1$$

from which it follows that $\lim_{x \to -2} f(x)$ does not exist.

Solution. (b)

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 5) = 0^2 - 5 = -5$$

Solution. (c)

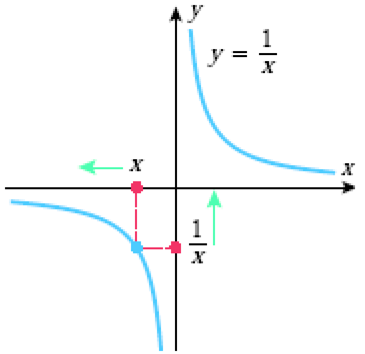
$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 5) = 3^2 - 5 = 4$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x+13} = \sqrt{\lim_{x \to 3^+} (x+13)} = \sqrt{3+13} = 4$$

$$\lim_{x \to 3} f(x) = 4$$

LIMITS AT INFINITY: END BEHAVIOR OF A FUNCTION

The behavior of a function *f(x) as x increases without bound or decreases without bound is sometimes called the* ***end behavior of the function.***

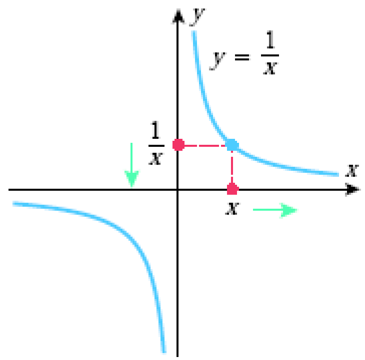


LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

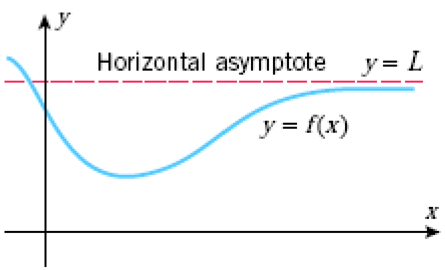
Example :

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{x} = 0$$

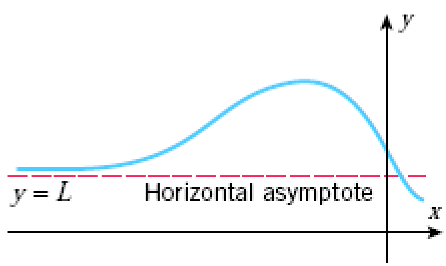
	VALUES						CONCLUSION
x $1/x$	-1 -1	-10 -0.1	-100 -0.01	-1000 -0.001	$-10{,}000$ -0.0001	$\cdots$ $\cdots$	As $x \to -\infty$ the value of $1/x$ increases toward zero.
x $1/x$	1 1	10 0.1	100 0.01	1000 0.001	$10{,}000$ 0.0001	$\cdots$ $\cdots$	As $x \to +\infty$ the value of $1/x$ decreases toward zero.



The end behavior of a function f *when*



$$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$



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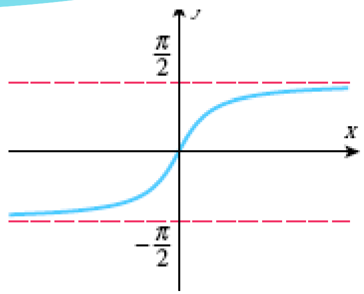
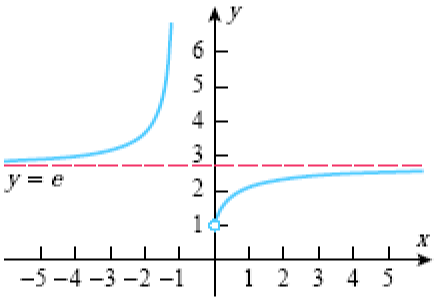
LIMITS OF PIECEWISE-DEFINED FUNCTIONS



$y = \tan^{-1} x$

Example $\lim_{x \to +\infty} \tan^{-1} x = \frac{\pi}{2}$ and $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

so the line $y = \pi/2$ is a horizontal asymptote for f in the positive direction and the line $y = -\pi/2$ is a horizontal asymptote in the negative direction.



$y = \left(1 + \frac{1}{x}\right)^x$

Example $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$ and $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$

so the line $y = e$ is a horizontal asymptote for f in both the positive and negative directions.

LIMIT LAWS FOR LIMITS AT INFINITY

$$\lim_{x \to +\infty} (f(x))^n = \left(\lim_{x \to +\infty} f(x)\right)^n \qquad \lim_{x \to -\infty} (f(x))^n = \left(\lim_{x \to -\infty} f(x)\right)^n$$

$$\lim_{x \to +\infty} kf(x) = k \lim_{x \to +\infty} f(x) \qquad \lim_{x \to -\infty} kf(x) = k \lim_{x \to -\infty} f(x)$$

n is a positive integer

$$\lim_{x \to +\infty} k = k \qquad \lim_{x \to -\infty} k = k$$

Examples

$$\lim_{x\to+\infty}\frac{1}{x^n}=\left(\lim_{x\to+\infty}\frac{1}{x}\right)^n=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x^n}=\left(\lim_{x\to-\infty}\frac{1}{x}\right)^n=0$$

n is a positive integer

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^x=\lim_{x\to+\infty}\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{1/2}$$

$$=\left[\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{2x}\right]^{1/2}=e^{1/2}=\sqrt{e}$$

INFINITE LIMITS AT INFINITY

If the values of *f(x)* increase without bound as

$x\to+\infty$ or as $x\to-\infty$, then we write

$$\lim_{x\to+\infty}f(x)=+\infty \quad \text{or} \quad \lim_{x\to-\infty}f(x)=+\infty$$

as appropriate; and if the values of $f(x)$ decrease without bound as $x\to+\infty$ or as $x\to-\infty$, then we write

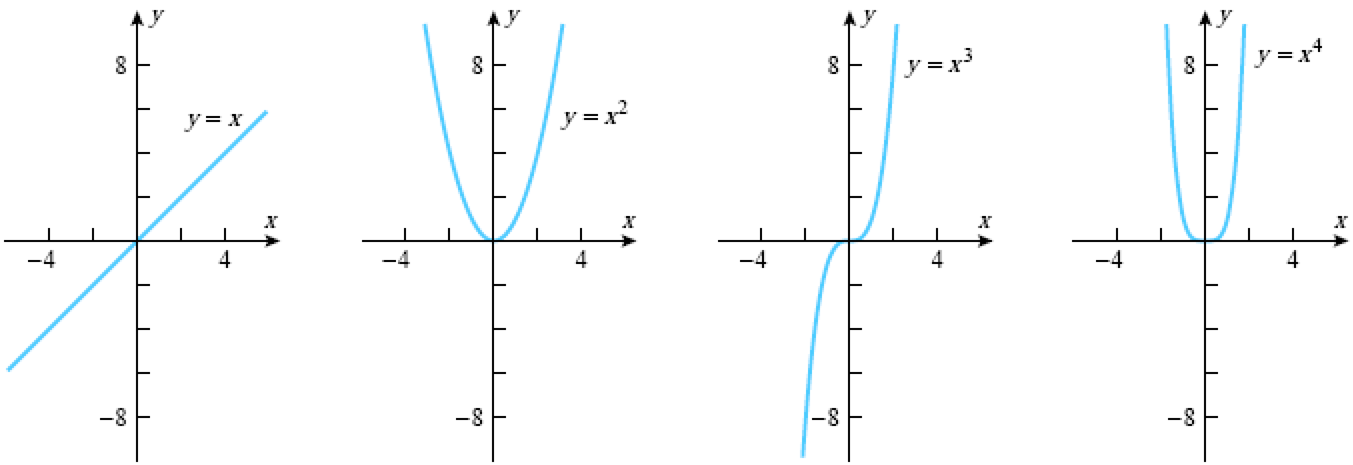
$$\lim_{x\to+\infty}f(x)=-\infty \quad \text{or} \quad \lim_{x\to-\infty}f(x)=-\infty$$

as appropriate.

LIMITS OF x^n AS $x \to \pm\infty$

$$\lim_{x \to +\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots$$

$$\lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases}$$



$$\lim_{x \to +\infty} x = +\infty \qquad \lim_{x \to -\infty} x = -\infty$$

$$\lim_{x \to +\infty} x^2 = +\infty \qquad \lim_{x \to -\infty} x^2 = +\infty$$

$$\lim_{x \to +\infty} x^3 = +\infty \qquad \lim_{x \to -\infty} x^3 = -\infty$$

$$\lim_{x \to +\infty} x^4 = +\infty \qquad \lim_{x \to -\infty} x^4 = +\infty$$

Examples

$$\lim_{x \to +\infty} 2x^5 = +\infty, \qquad \lim_{x \to -\infty} 2x^5 = -\infty$$

$$\lim_{x \to +\infty} -7x^6 = -\infty, \qquad \lim_{x \to -\infty} -7x^6 = -\infty$$

LIMITS OF POLYNOMIALS AS $x \to \pm\infty$

The end behavior of a polynomial matches the end behavior of its highest degree term.

More precisely, if $c_n \neq 0$, then

$$\lim_{x \to -\infty} (c_0 + c_1 x + \cdots + c_n x^n) = \lim_{x \to -\infty} c_n x^n$$

$$\lim_{x \to +\infty} (c_0 + c_1 x + \cdots + c_n x^n) = \lim_{x \to +\infty} c_n x^n$$

Examples

$$\lim_{x \to -\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x \to -\infty} 7x^5 = -\infty$$

$$\lim_{x \to -\infty} (-4x^8 + 17x^3 - 5x + 1) = \lim_{x \to -\infty} -4x^8 = -\infty$$

LIMITS OF RATIONAL FUNCTIONS AS $x \to \pm\infty$

Example Find $\displaystyle\lim_{x \to +\infty} \frac{3x + 5}{6x - 8}$.

Solution.

$$\lim_{x \to +\infty} \frac{3x+5}{6x-8} = \lim_{x \to +\infty} \frac{3 + \frac{5}{x}}{6 - \frac{8}{x}} = \frac{\lim\limits_{x \to +\infty} \left(3 + \frac{5}{x}\right)}{\lim\limits_{x \to +\infty} \left(6 - \frac{8}{x}\right)} = \frac{\lim\limits_{x \to +\infty} 3 + \lim\limits_{x \to +\infty} \frac{5}{x}}{\lim\limits_{x \to +\infty} 6 - \lim\limits_{x \to +\infty} \frac{8}{x}} = \frac{3 + 5 \lim\limits_{x \to +\infty} \frac{1}{x}}{6 - 8 \lim\limits_{x \to +\infty} \frac{1}{x}} = \frac{3+0}{6-0} = \frac{1}{2}$$

A QUICK METHOD FOR FINDING LIMITS OF RATIONAL FUNCTIONS AS $x \to +\infty$ OR $x \to -\infty$

The end behavior of a rational function matches the end behavior of the quotient of the highest degree term in the numerator divided by the highest degree term in the denominator.

Examples

$$\lim_{x\to+\infty}\frac{3x+5}{6x-8}=\lim_{x\to+\infty}\frac{3x}{6x}=\lim_{x\to+\infty}\frac{1}{2}=\frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4x^2}{2x^3}=\lim_{x\to-\infty}\frac{2}{x}=0$$

$$\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^3}{(-3x)}=\lim_{x\to+\infty}\left(-\frac{5}{3}x^2\right)=-\infty$$

LIMITS INVOLVING RADICALS

Example

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$$

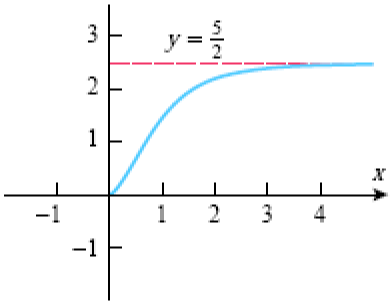
Solution.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to-\infty}\frac{\dfrac{\sqrt{x^2+2}}{|x|}}{\dfrac{3x-6}{|x|}}=\lim_{x\to-\infty}\frac{\dfrac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\dfrac{3x-6}{(-x)}}=\lim_{x\to-\infty}\frac{\sqrt{1+\dfrac{2}{x^2}}}{-3+\dfrac{6}{x}}=-\frac{1}{3}$$

$$\sqrt{x^2}=|x|.$$

Example

$$\lim_{x \to +\infty} (\sqrt{x^6 + 5x^3} - x^3)$$



$y = \sqrt{x^6 + 5x^3} - x^3, x \geq 0$

Solution.

$$\lim_{x \to +\infty} (\sqrt{x^6 + 5x^3} - x^3) = \lim_{x \to +\infty} (\sqrt{x^6 + 5x^3} - x^3) \left(\frac{\sqrt{x^6 + 5x^3} + x^3}{\sqrt{x^6 + 5x^3} + x^3} \right)$$

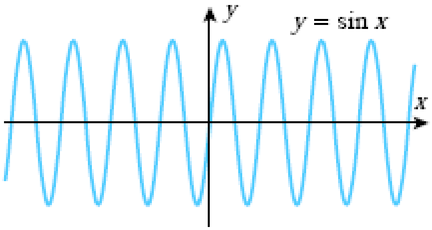
$$= \lim_{x \to +\infty} \frac{(x^6 + 5x^3) - x^6}{\sqrt{x^6 + 5x^3} + x^3} = \lim_{x \to +\infty} \frac{5x^3}{\sqrt{x^6 + 5x^3} + x^3}$$

$$= \lim_{x \to +\infty} \frac{5}{\sqrt{1 + \dfrac{5}{x^3}} + 1}$$

$\boxed{\sqrt{x^6} = x^3 \text{ for } x > 0}$

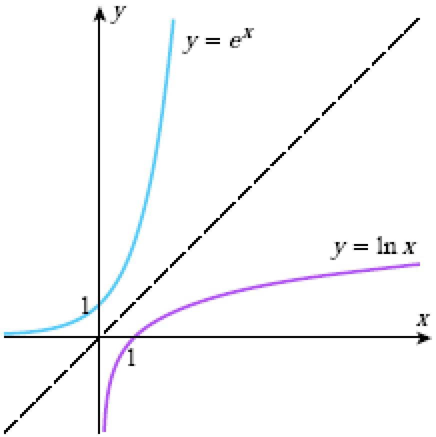
$$= \frac{5}{\sqrt{1 + 0} + 1} = \frac{5}{2}$$

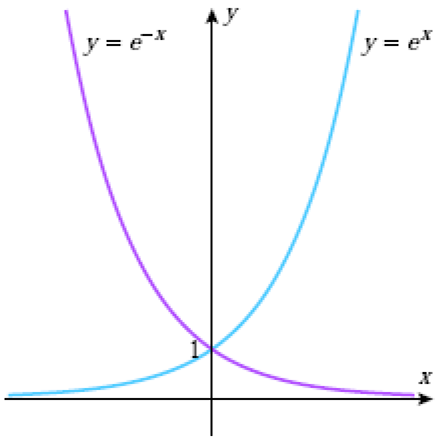
END BEHAVIOR OF TRIGONOMETRIC, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS



There is no limit as $x \to +\infty$ or $x \to -\infty$.

$$\lim_{x \to +\infty} \ln x = +\infty \qquad \lim_{x \to +\infty} e^x = +\infty$$



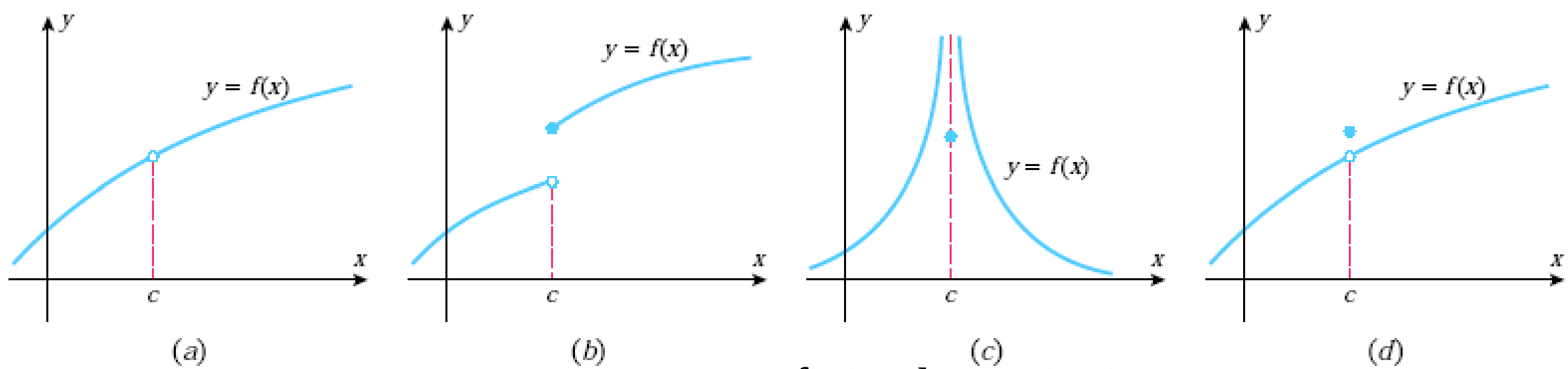


$$\lim_{x \to +\infty} e^{-x} = 0 \qquad \lim_{x \to -\infty} e^{-x} = +\infty$$

CONTINUITY

Definition: A function *f is said to be continuous at $x = c$ provided the* following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.



function is not defined at c.

jump discontinuity at c

Infinite discontinuity at c.

removable discontinuity at c.

Example

Determine whether the following functions are continuous at *x = 2.*

$$f(x) = \frac{x^2-4}{x-2}, \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

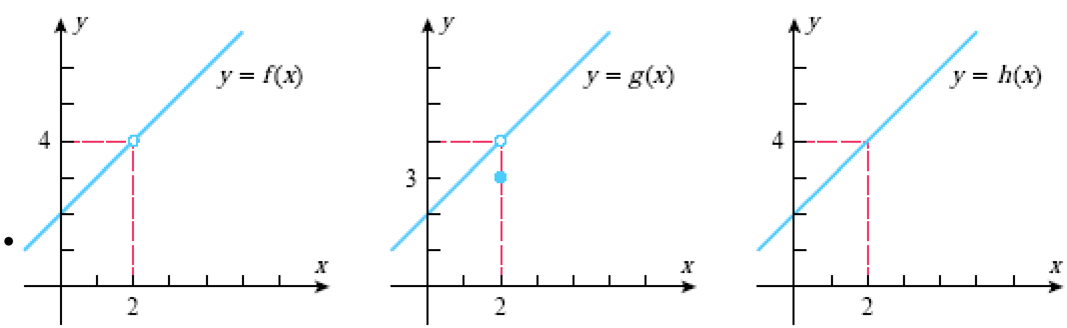
Solution.
$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 4$$

The function *f is undefined at x = 2, and hence is not continuous at x = 2*

The function *g is defined at x = 2, but its value there is g(2) = 3, which is not the same as* the limit as *x approaches 2; hence, g is also not continuous at x = 2*

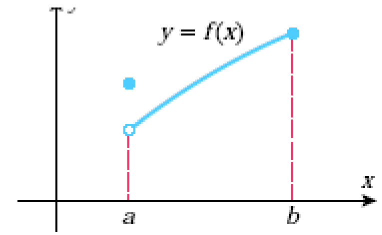
The value of the function *h at x = 2 is h(2) = 4, which is the same as the limit as x approaches* 2; hence, *h is continuous at x = 2*

***h(2) = 4, which is same as the limit as x approaches* 2;**
hence, *h is continuous at x = 2* .
We can write h(x) = x + 2.



Definition: A function *f is said to be continuous on a closed interval [a, b]* **if the following conditions are satisfied:**

1. ***f is continuous on (a, b).***
2. ***f is continuous from the right at a.***
3. ***f is continuous from the left at b.***



f **is Continuous at the right endpoint of [*a, b] because*** $\lim_{x \to b^-} f(x) = f(b)$

It is not continuous at the left endpoint because $\lim_{x \to a^+} f(x) \neq f(a)$

f is continuous from the left at c if $\lim_{x \to c^-} f(x) = f(c)$

f is continuous from the right at c if $\lim_{x \to c^+} f(x) = f(c)$

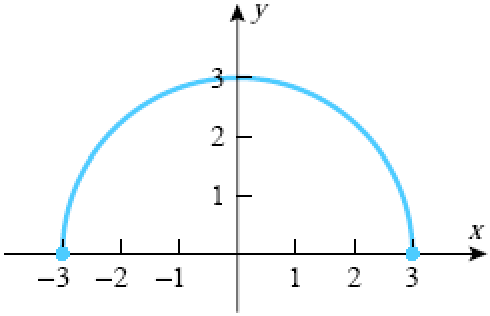
Example: What can you say about the continuity of the function

Solution. $f(x) = \sqrt{9 - x^2}$ If c is any point in the interval $(-3, 3)$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$



Thus, f is continuous on the closed interval $[-3, 3]$

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

Theorem: *If the functions f and g are continuous at c, then*
(a) $f + g$ is continuous at c.
(b) $f - g$ is continuous at c.
(c) $f\,g$ is continuous at c.
(d) f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.

CONTINUITY OF POLYNOMIALS AND RATIONAL FUNCTIONS

If *p(x) is a polynomial and a is any real number, then*

$$\lim_{x \to a} p(x) = p(a)$$

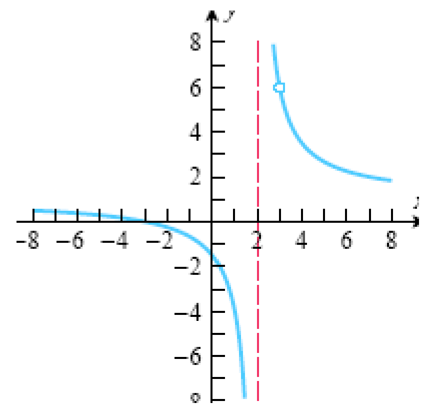
Theorem:
(a) A polynomial is continuous everywhere.
(b) A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.

Example: **For what values of *x is there a discontinuity in the graph of***

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. **The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation**

$$x^2 - 5x + 6 = 0$$



yields discontinuities at x = 2 and at x = 3

Example: Show that $|x|$ is continuous everywhere

Solution.

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

The polynomial *x on (0,+∝) and is the same as the polynomial −x on (− ∝ , 0). But polynomials are continuous everywhere, so x = 0 is the only possible discontinuity for*

$$\lim_{x \to 0} |x| = 0 \quad |0| = 0$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

CONTINUITY OF COMPOSITIONS

Theorem: *If* $\lim_{x\to c} g(x) = L$ *and if the function* f *is continuous at* L, *then*

$\lim_{x\to c} f(g(x)) = f(L)$. *That is,*

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$

This equality remains valid if $\lim_{x\to c}$ *is replaced everywhere by one of* $\lim_{x\to c^+}$, $\lim_{x\to c^-}$, $\lim_{x\to +\infty}$, *or* $\lim_{x\to -\infty}$.

Example:

$$\lim_{x\to c} |g(x)| = \left|\lim_{x\to c} g(x)\right|$$

provided $\lim_{x\to c} g(x)$ exists. Thus, for example,

$$\lim_{x\to 3} |5 - x^2| = \left|\lim_{x\to 3} (5 - x^2)\right| = |-4| = 4$$

Theorem:

(a) ***If the function g is continuous at c, and the function f is continuous at g(c), then the composition $f \circ g$ is continuous at c.***

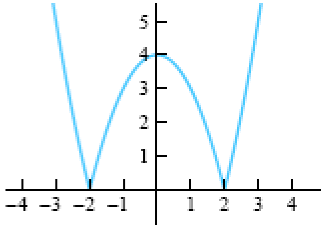
(b) ***If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.***

The absolute value of a continuous function is continuous.

Example:

$g(x) = 4 - x^2$ is continuous everywhere, so

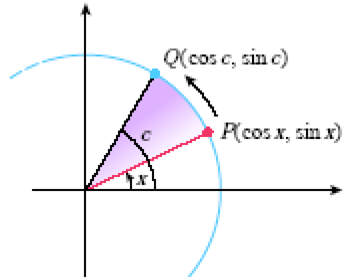
$|4 - x^2|$ is also continuous everywhere.



CONTINUITY OF TRIGONOMETRIC, EXPONENTIAL, AND INVERSE FUNCTIONS:

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c$$

$$\lim_{x \to c} \tan x = \lim_{x \to c} \frac{\sin x}{\cos x} = \frac{\sin c}{\cos c} = \tan c \qquad \cos c \neq 0$$



Theorem: ***If c is any number in the natural domain of the stated trigonometric function, then***

$$\lim_{x \to c} \sin x = \sin c \qquad \lim_{x \to c} \cos x = \cos c \qquad \lim_{x \to c} \tan x = \tan c$$

$$\lim_{x \to c} \csc x = \csc c \qquad \lim_{x \to c} \sec x = \sec c \qquad \lim_{x \to c} \cot x = \cot c$$

Example: Find the limit

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right)$$

Solution.

Since the cosine function is continuous everywhere, it follows from Theorem

$$\lim_{x \to 1} \cos(g(x)) = \cos\left(\lim_{x \to 1} g(x)\right)$$

provided $\lim_{x \to 1} g(x)$ exists. Thus,

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right) = \lim_{x \to 1} \cos(x + 1) = \cos\left(\lim_{x \to 1} (x + 1)\right) = \cos 2$$

CONTINUITY OF INVERSE FUNCTIONS

Theorem: ***If f is a one-to-one function that is continuous at each point of its domain, then f^{-1} is continuous at each point of its domain; that is, f^{-1} is continuous at each point in the range of f .***

Example: **Use Theorem to prove that** $\sin^{-1} x$ ***is continuous on the interval*** **[−1, 1]**.

Solution. $\sin^{-1} x$ ***is the inverse of the restricted sine function whose domain*** **is the interval [*−π/2, π/2] and whose range is the interval [−1, 1]*. Since sin *x is continuous on the interval [−π/2, π/2], Theorem implies*** $\sin^{-1} x$ ***is continuous on the interval [−1, 1].***

Theorem: *Let $b > 0, b \neq 1$.*

(a) *The function b^x is continuous on $(-\infty, +\infty)$.*

(b) *The function $\log_b x$ is continuous on $(0, +\infty)$.*

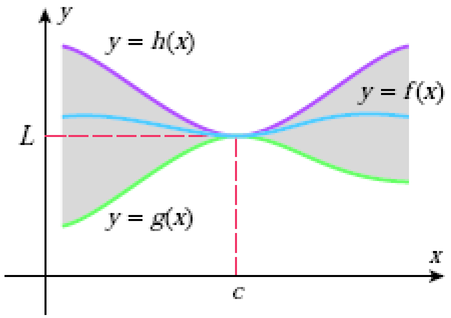
Example: **Where is the function** $f(x) = \dfrac{\tan^{-1} x + \ln x}{x^2 - 4}$ ***continuous.***

Solution.

The fraction will be continuous at all points where the numerator and denominator are both continuous and the denominator is nonzero.

Since $\tan^{-1} x$ ***is continuous*** **everywhere and ln** ***x is continuous if x > 0, the numerator is continuous if x > 0. The*** **denominator is continuous everywhere, so the fraction will be continuous at** ***x > 0 and the denominator is nonzero. Thus, f is continuous on (0, 2) and (2,*** $+\infty$***).***

Theorem:(*The Squeezing Theorem)*



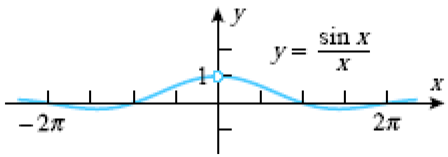
Let f , g, and h be functions satisfying

$$g(x) \le f(x) \le h(x)$$

for all x in some open interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c,say $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

then f also has this limit as x approaches c, that is, $\lim_{x \to c} f(x) = L$

Example: $\lim_{x \to 0} \frac{\sin x}{x} = 1$



Solution.

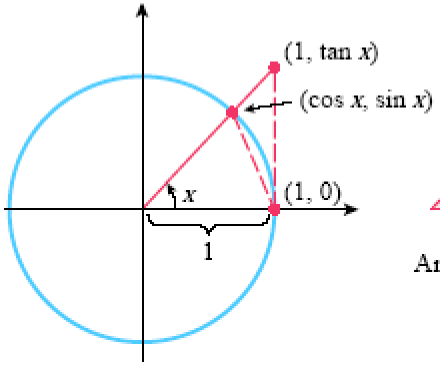
$$\frac{1}{2}\tan x \ge \frac{1}{2}x \ge \frac{1}{2}\sin x$$

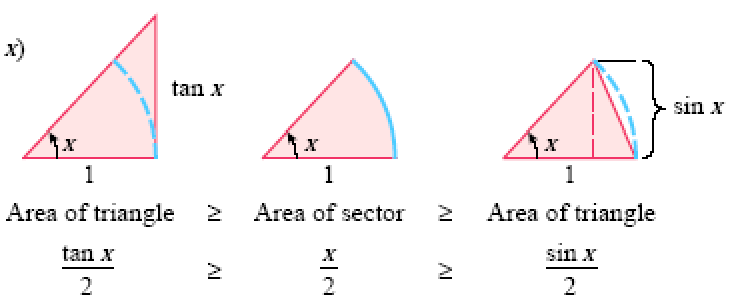
$$\frac{1}{\cos x} \ge \frac{x}{\sin x} \ge 1$$

$$\cos x \le \frac{\sin x}{x} \le 1$$

$$\lim_{x \to 0} \cos x = 1 \quad \text{and} \quad \lim_{x \to 0} 1 = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

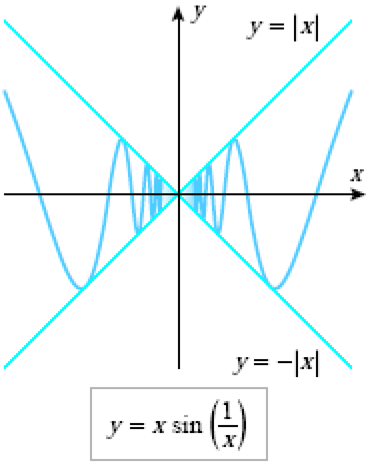




Example: $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

Solution.

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$



it follows that if $x \neq 0$, then

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$$

Since $|x| \to 0$ as $x \to 0$,

The inequalities and Theorem imply that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$$