COMSATS University Islamabad

Islamabad Campus

Semester – Spring 2021

MTH104 Calculus and Analytic Geometry **Instructor: Dr. M. Yousaf**

Lecture #	Topics
1	Real numbers, Inequalities
2	Functions, domain and range of functions, composition of functions
3	Translation, reflection, stretches and compressions of functions, symmetry tests, polynomials, rational functions , algebraic functions
4	Inverse functions(trigonometric), exponential and logarithmic functions, parametric equations
5	Concept of limits, Computing limits
6	Limits at infinity, Continuity,
7	Slope of the tangent line, General rates of change, the derivative function,
8	Techniques of differentiation, Chain Rule
9	Implicit differentiation, Differentiation of inverse trigonometric functions,
10	L' Hopital rule
11	Sessional 1

12	Increasing and decreasing functions, concavity, Inflection points
13	Relative maxima and minima, First and second derivative tests
14	Applied maximum and minimum problems
15	Rolle's theorem, Mean value theorem, The indefinite integral
16	Integration by substitution, Integration by parts, Integration of rational functions by partial fractions
17	Definition of area as a limit; sigma notation,
18	Definition of area as a limit; sigma notation
19	Riemann sums and the definite integrals, Fundamental theorem of calculus, MVT for integral
20	Applications of definite integrals: Area between two curves
21	Sessional II

22	Applications of definite integrals: Volume by slicing, volume by disk method, volume by washers
23	Applications of definite integrals: Volume by cylindrical shells
24	Applications of definite integrals: Length of a plane curve
25	Sequences, Monotone sequences
26	The divergence test, integral test, p-series test
27	The comparison , ratio and root tests
28	Alternating series; Conditional convergence
29	Maclaurin and Taylor Polynomials, Maclaurin and Taylor series
30	Power series , Convergence of Taylor series

Reading Materials

Text book: **CALCULUS** By Howard **Anton** [10th Edition] John Wiley and Sons, Inc (International Edition)

Additional reading:

- **Advanced Engineering Mathematics**,Kreyszig, E, Wiley and Sons.
- **CALCULUS** by Earl W. Swokowsky,[6th Edition], PWS Publishing Co.
- Calculus by **James Stewart,** [Second Edition].
- Calculus by Thomas and Finny

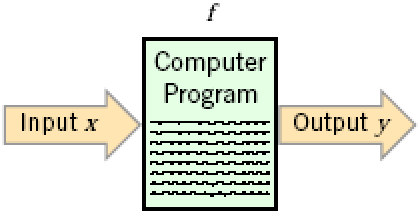
Assessment Scheme

Quizes	15
Assignments	10
Sessional – I	10
Sessional – II	15
Terminal	50
Total	100

Attendance Requirement : At least 80%

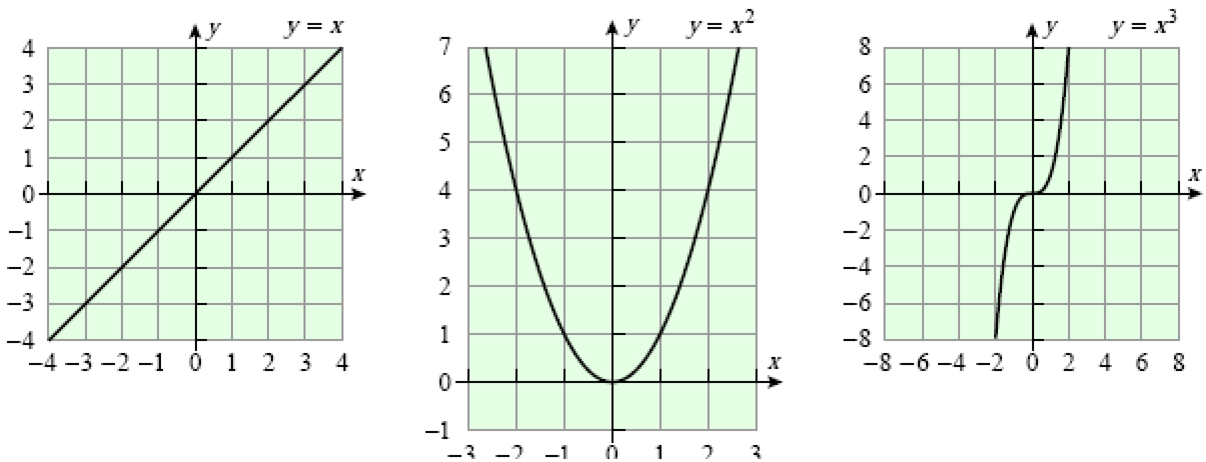
Function

A *function f is a rule that associates a unique output with each* input. If the input is denoted by *x, then the output is denoted by f (x).*



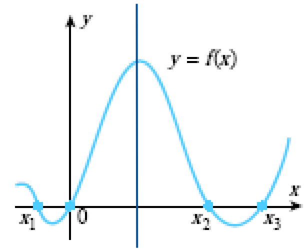
The equation $y = f(x)$ expresses *y as a function of x; the variable x is called the independent variable (or argument) of f , and the variable y is called the dependent variable of f*

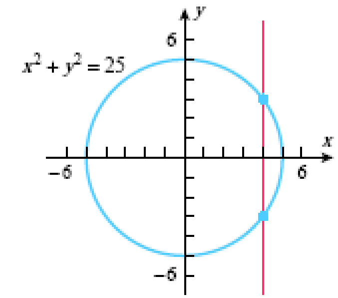
GRAPHS OF FUNCTIONS



The vertical line test

A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.





THE ABSOLUTE VALUE FUNCTION

The ***absolute value or magnitude of a real number x is defined by***

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|5| = 5, \quad \left|-\tfrac{4}{7}\right| = \tfrac{4}{7}, \quad |0| = 0$$

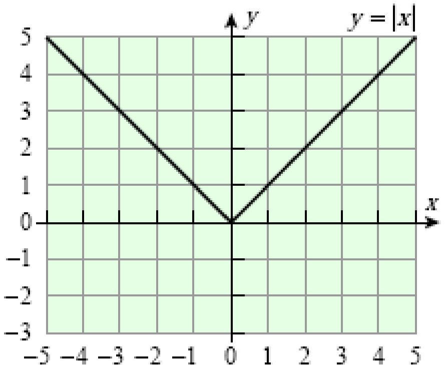
properties of absolute value

(a)	$\lvert -a\rvert = \lvert a\rvert$	A number and its negative have the same absolute value.
(b)	$\lvert ab\rvert = \lvert a\rvert\,\lvert b\rvert$	The absolute value of a product is the product of the absolute values.
(c)	$\lvert a/b\rvert = \lvert a\rvert/\lvert b\rvert,\ b \neq 0$	The absolute value of a ratio is the ratio of the absolute values.
(d)	$\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$	The *triangle inequality*

A statement that is correct for all real values of *x is* $\sqrt{x^2} = |x|$

PIECEWISE-DEFINED FUNCTIONS

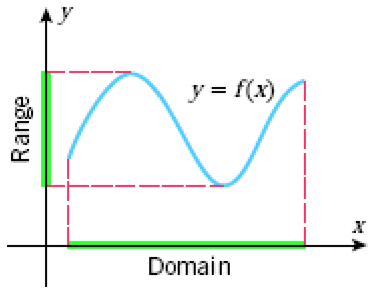
$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



DOMAIN AND RANGE

If *x and y are related by the equation y = f(x), then the set of all allowable inputs (x-values)* is called the *domain of f , and the set of outputs (y-values) that result when x varies over* the domain is called the *range of f.*

If no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain of the function.***



Example

Find the natural domain of (a) $f(x) = x^3$ (b) $f(x) = 1/[(x-1)(x-3)]$

Solution (a). The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

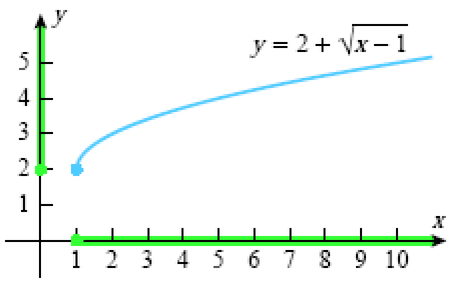
Example

Find the domain and range of

(a) $f(x) = 2 + \sqrt{x-1}$ (b) $f(x) = (x+1)/(x-1)$

Solution (a). Since no domain is stated explicitly, the domain of f is its natural domain,

$[1, +\infty)$ As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f.



Solution (b). The given function f is defined for all real x, except x = 1, so the natural domain of *f is*

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$

To determine the range $y = \dfrac{x+1}{x-1}$

we solve for *x in terms of y*

$$(x-1)y = x+1$$
$$xy - y = x + 1$$

$$xy - x = y + 1$$
$$x(y-1) = y + 1$$
$$x = \frac{y+1}{y-1}$$

It is clear from the equation that y = 1 is not in the range . So the range is

$$\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$

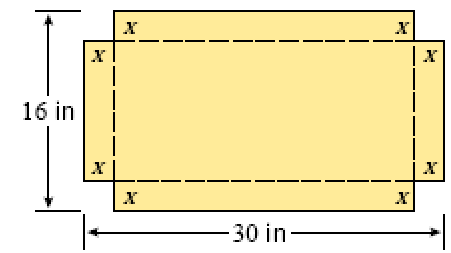
Example : An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides.

(a) Let *V be the volume of the box that results when the squares have sides of length x.* Find a formula for *V as a function of x.*

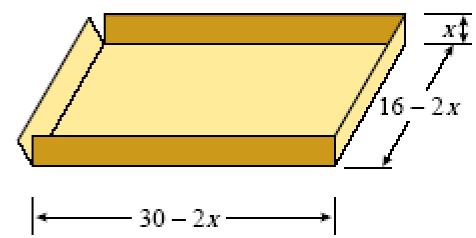
(b) Find the domain of *V* .

(c) Use the graph of *V to estimate the range of V* .

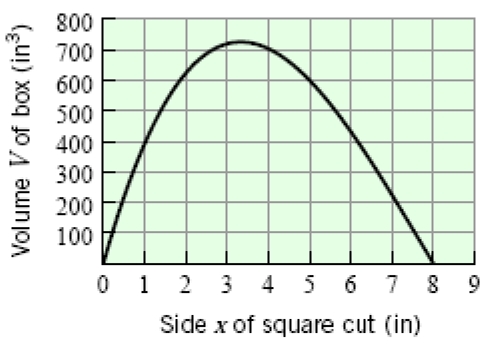
(d) Describe in words what the graph tells you about the volume



Solution (a). $V(x) = (16-2x)(30-2x)x = 480x - 92x^2 + 4x^3$



Solution (b). $0 \le x \le 8$



Solution (c). $0 \le V \le 725$

Solution (d).

Max. volume occurs for a value of *x* between 3 and 4

NEW FUNCTIONS FROM OLD

Definition: Given functions *f and g, we define*

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(f/g)(x) = f(x)/g(x)$$

For the functions $f+g$, $f-g$, and fg we define the domain to be the intersection of the domains of f and g, and for the function f/g we define the domain to be the intersection of the domains of f and g but with the points where $g(x) = 0$ excluded (to avoid division by zero).

Example:

Let

$$f(x) = 1 + \sqrt{x-2} \quad \text{and} \quad g(x) = x - 3$$

Find the domains and formulas for the functions $f+g$, $f-g$, fg, f/g

Solution.

$$(f+g)(x) = f(x) + g(x) = (1 + \sqrt{x-2}) + (x-3) = x - 2 + \sqrt{x-2}$$
$$(f-g)(x) = f(x) - g(x) = (1 + \sqrt{x-2}) - (x-3) = 4 - x + \sqrt{x-2}$$
$$(fg)(x) = f(x)g(x) = (1 + \sqrt{x-2})(x-3)$$
$$(f/g)(x) = f(x)/g(x) = \frac{1 + \sqrt{x-2}}{x-3}$$

The domains of f and g are $[2, +\infty)$ and $(-\infty, +\infty)$, respectively (their natural domains).

The intersection of these two domains is $[2, +\infty) \cap (-\infty, +\infty) = [2, +\infty)$

Moreover, since $g(x) = 0$ if $x = 3$, the domain of f/g $[2, 3) \cup (3, +\infty)$

COMPOSITION OF FUNCTIONS

Definition: Given functions *f and g, the composition of f with g, denoted* by *f* $\circ$*g, is the function defined by (f* $\circ$*g)(x) = f(g(x))* The domain of *f* $\circ$*g is defined to consist of all x in the domain of g for which g(x) is* in the domain of *f* .

Example: Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

$$(f \circ g)(x)$$

Solution The formula for *f(g(x)) is*

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x}\,)^2 + 3 = x + 3$$

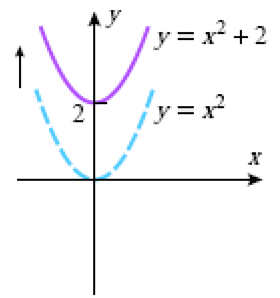
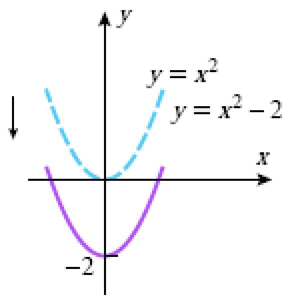
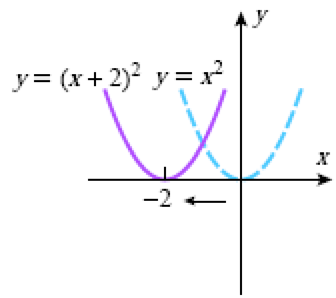
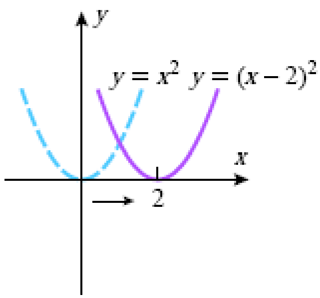
Since the domain of g is $[0, +\infty)$ and the domain of f is $(-\infty, +\infty)$, the domain of $f \circ g$ consists of all x in $[0, +\infty)$ such that $g(x) = \sqrt{x}$ lies in $(-\infty, +\infty)$; thus, the domain of $f \circ g$ is $[0, +\infty)$. Therefore,

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

TRANSLATIONS

The geometric effect on the graph of $y = f(x)$ *of adding or subtracting* a *positive constant c to f or to its independent variable x*.

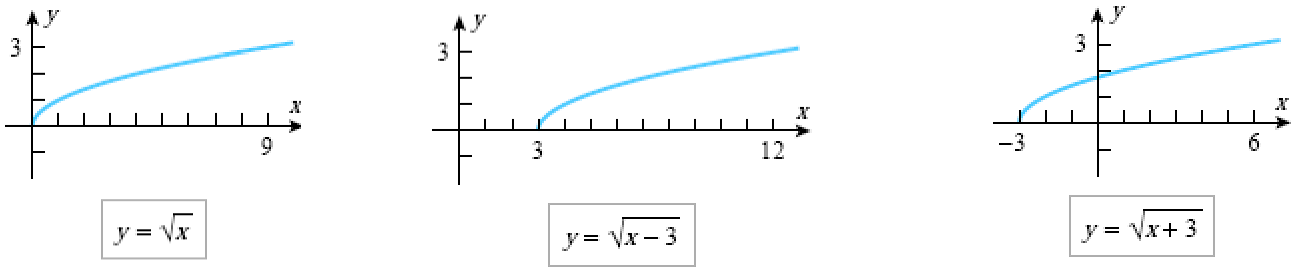
TRANSLATION PRINCIPLES

OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				

Example: **Sketch the graph of**

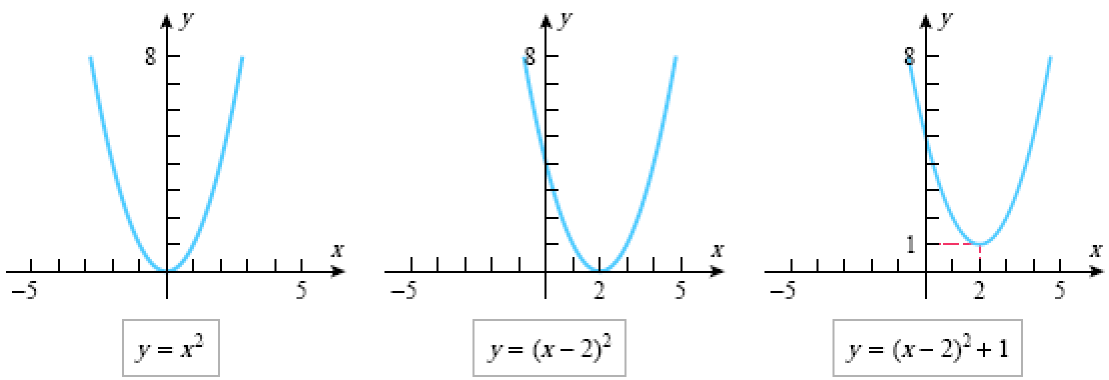
(a) $y = \sqrt{x-3}$ (b) $y = \sqrt{x+3}$

Solution.



Example: **Sketch the graph of** $y = x^2 - 4x + 5.$

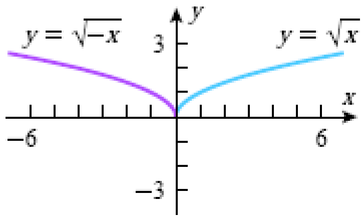
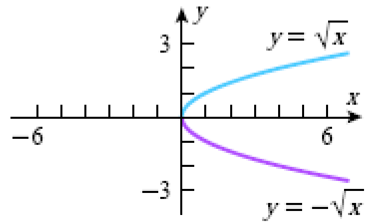
Solution. $y = (x^2 - 4x + 4) - 4 + 5 = (x-2)^2 + 1$



REFLECTIONS

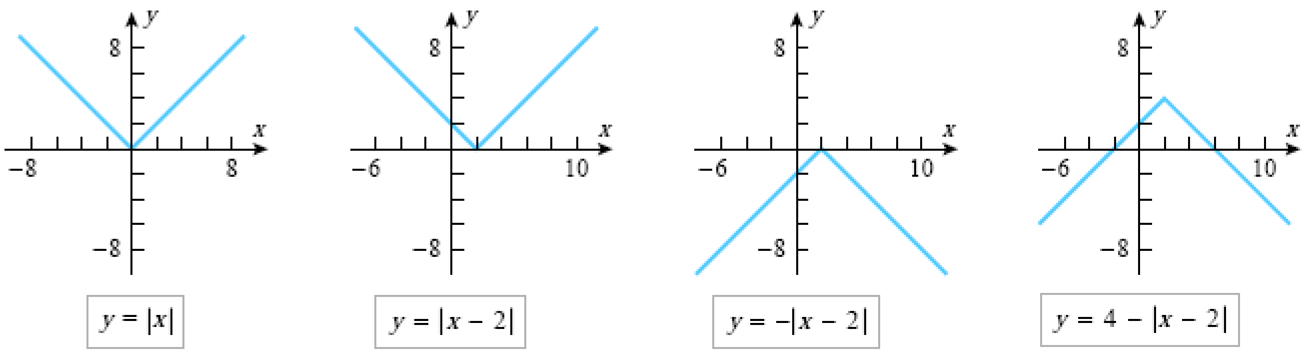
The graph of *$y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis* because the point *(x, y) on the graph of $f(x)$ is replaced by $(-x, y)$. Similarly, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(x, -y)$ [the equation $y = -f(x)$ is equivalent* to *$-y = f(x)$].*

REFLECTION PRINCIPLES

OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
EXAMPLE		

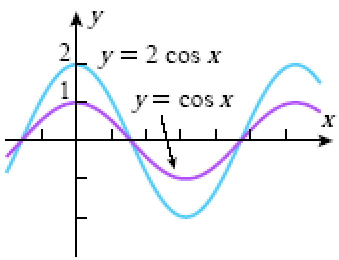
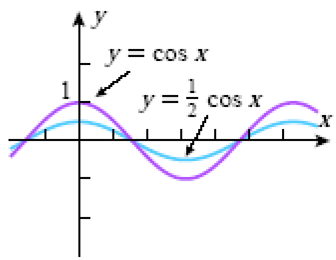
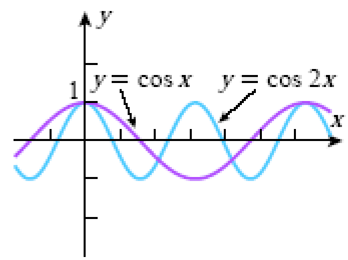
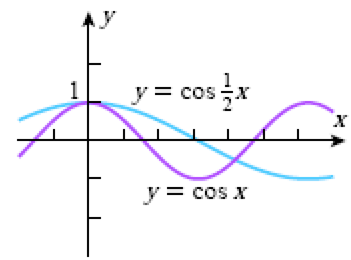
Example: **Sketch the graph of** $y = 4 - |x - 2|$.

Solution.

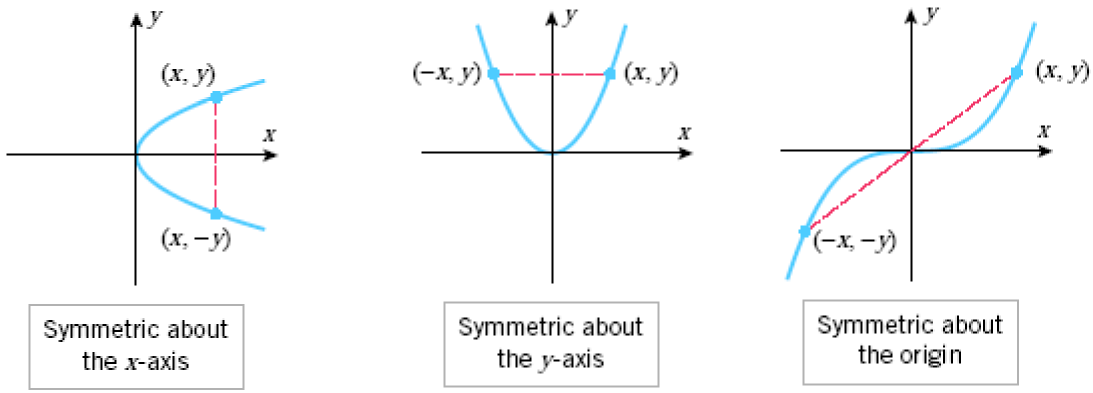


STRETCHES AND COMPRESSIONS

STRETCHING AND COMPRESSING PRINCIPLES

OPERATION ON $y = f(x)$	Multiply $f(x)$ by c $(c > 1)$	Multiply $f(x)$ by c $(0 < c < 1)$	Multiply x by c $(c > 1)$	Multiply x by c $(0 < c < 1)$
NEW EQUATION	$y = cf(x)$	$y = cf(x)$	$y = f(cx)$	$y = f(cx)$
GEOMETRIC EFFECT	Stretches the graph of $y = f(x)$ vertically by a factor of c	Compresses the graph of $y = f(x)$ vertically by a factor of $1/c$	Compresses the graph of $y = f(x)$ horizontally by a factor of c	Stretches the graph of $y = f(x)$ horizontally by a factor of $1/c$
EXAMPLE				

SYMMETRY



Theorem (*Symmetry Tests*)

(a) A plane curve is symmetric about the y-axis if and only if replacing x by −x in its equation produces an equivalent equation.

(b) A plane curve is symmetric about the x-axis if and only if replacing y by −y in its equation produces an equivalent equation.

(c) A plane curve is symmetric about the origin if and only if replacing both x by −x and y by −y in its equation produces an equivalent equation.

Example: **Use Theorem to identify symmetries in the graph of** $x = y^2$.

Solution. Replacing y by $-y$ yields $x = (-y)^2$, which simplifies to the original equation $x = y^2$. Thus, the graph is symmetric about the x-axis. The graph is not symmetric about the y-axis because replacing x by $-x$ yields $-x = y^2$, which is not equivalent to the original equation $x = y^2$. Similarly, the graph is not symmetric about the origin because replacing x by $-x$ and y by $-y$ yields $-x = (-y)^2$, which simplifies to $-x = y^2$, and this is again not equivalent to the original equation.