Using *x as the* **independent variable for both** f and f^{-1}

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in the domain of } f^{-1}$$

Example: Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$
$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$
$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x$$

Example: Find a formula for the inverse of $f(x) = \sqrt{3x-2}$ with x as the independent variable, and state the domain of f^{-1}.

Solution

$$y = \sqrt{3x-2}$$

Solve this equation for ***x as a function of y***

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2 + 2)$$

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2)$$

$$f^{-1}(x) = \tfrac{1}{3}(x^2 + 2)$$

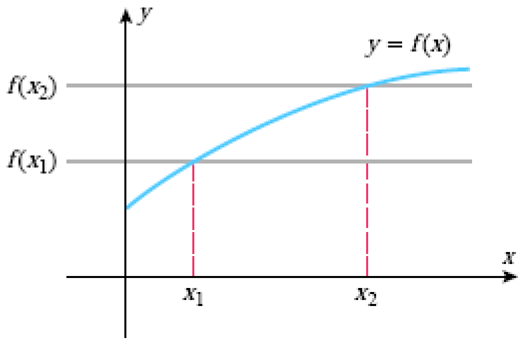
The natural domain of above equation is $(-\infty, +\infty)$**,** ***whereas the range of f(x) is*** $[0, +\infty)$.

domain of f^{-1} is the range of f.

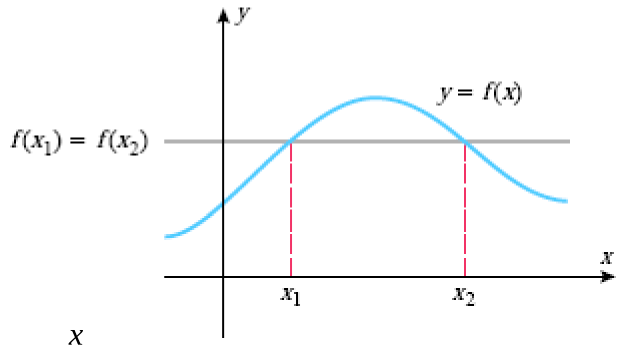
$$f^{-1}(x) = \tfrac{1}{3}(x^2 + 2), \quad x \geq 0$$

Theorem : ***A function has an inverse if and only if it is one-to-one.***

Theorem: (*The Horizontal Line Test) A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*



One-to-one, since $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

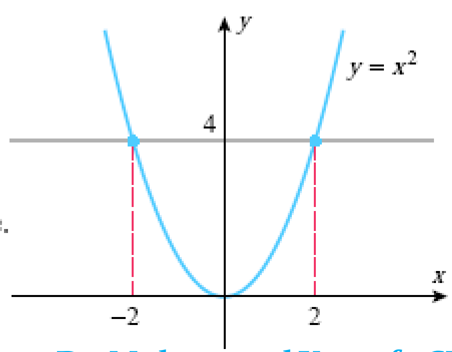


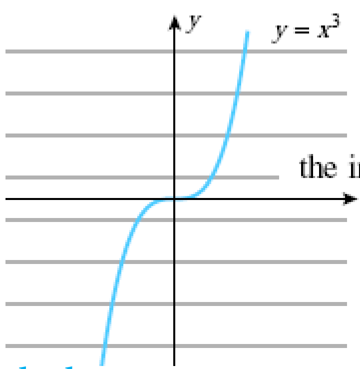
x

Not one-to-one, since $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

Example: **Use the horizontal line test to show that *f(x)* = x^2 *has no inverse but that f(x)* = x^3 *does.***

$f(x) = x^2$ is not invertible.





the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$

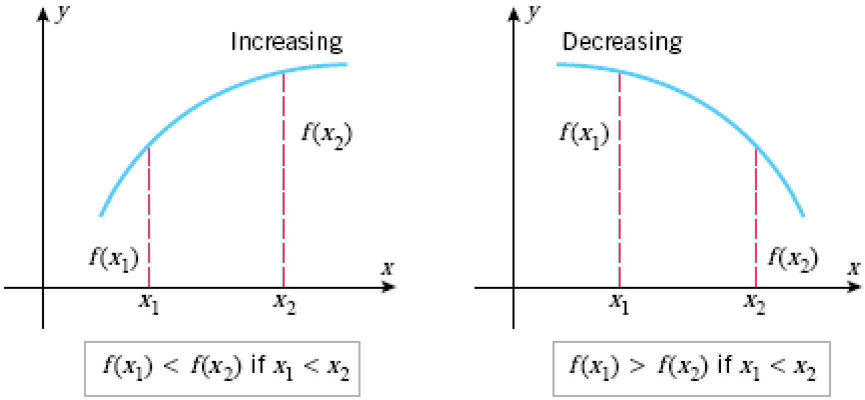
INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

If x_1 and x_2 are points in the domain of a function f, then f is increasing if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2$$

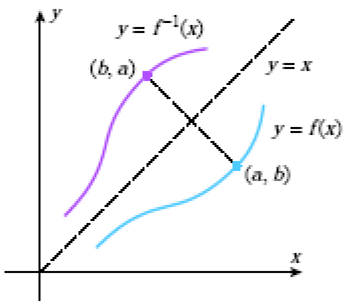
and f is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2$$



GRAPHS OF INVERSE FUNCTIONS

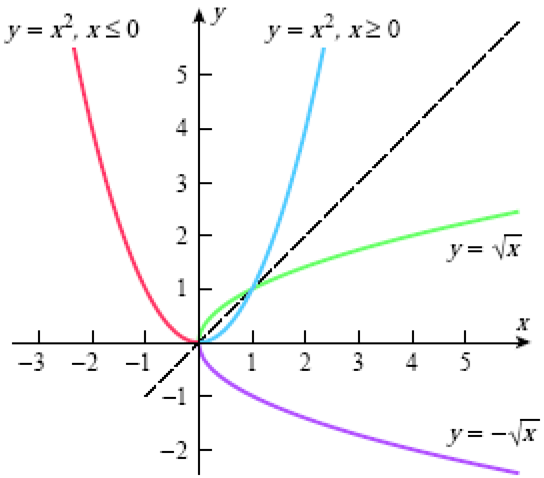
Theorem: *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*



RESTRICTING DOMAINS FOR INVERTIBILITY

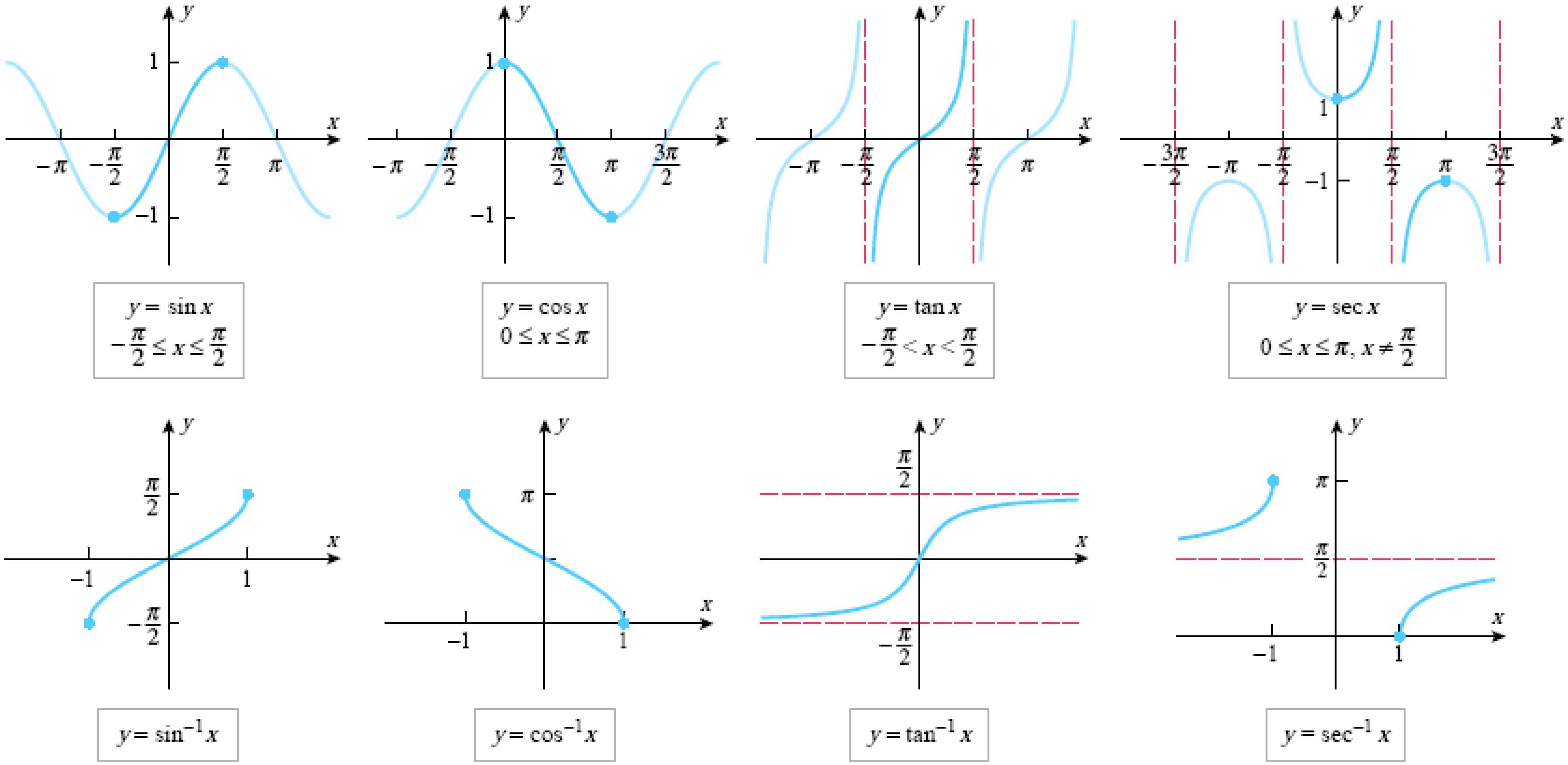
$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

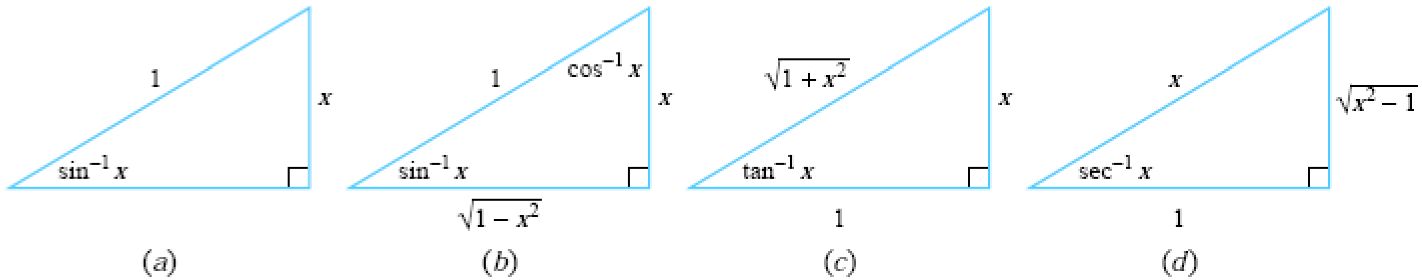
$$f_1^{-1}(x) = \sqrt{x} \quad \text{and} \quad f_2^{-1}(x) = -\sqrt{x}$$



x^2

INVERSE TRIGONOMETRIC FUNCTIONS





FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi$, $x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x \rvert \ge 1$

identities involving inverse trigonometric functions that are valid for $-1 \leq x \leq 1$;

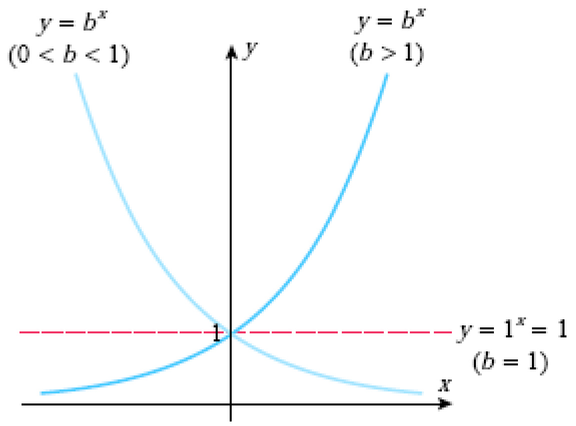
$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

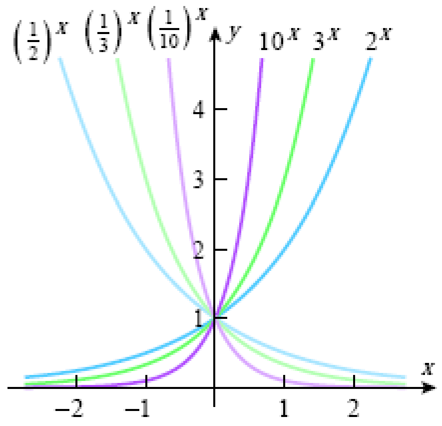
$$\cos(\sin^{-1} x) = \sqrt{1 - x^2}$$

$$\sin(\cos^{-1} x) = \sqrt{1 - x^2}$$

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}$$

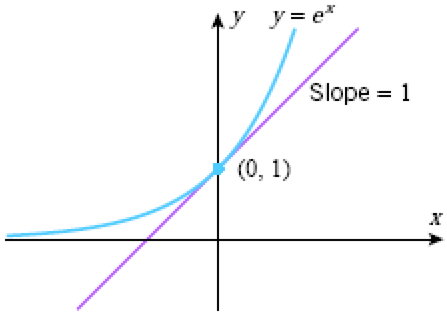
EXPONENTIAL AND LOGARITHMIC FUNCTIONS





THE NATURAL EXPONENTIAL FUNCTION

The function $f(x) = e^x$ is called the ***natural exponential function***. $e \approx 2.718282$



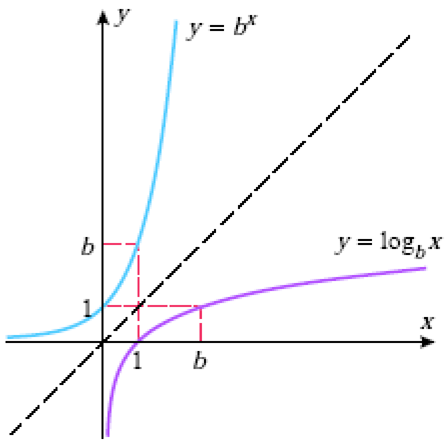
APPROXIMATIONS OF e BY $(1 + 1/x)^x$ FOR INCREASING VALUES OF x

x	$1 + \frac{1}{x}$	$\left(1 + \frac{1}{x}\right)^x$
1	2	≈ 2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

LOGARITHMIC FUNCTIONS

$$f(x) = \log_b x$$

THEOREM *If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.*



$y = \ln x$ if and only if $x = e^y$

CORRESPONDENCE BETWEEN PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

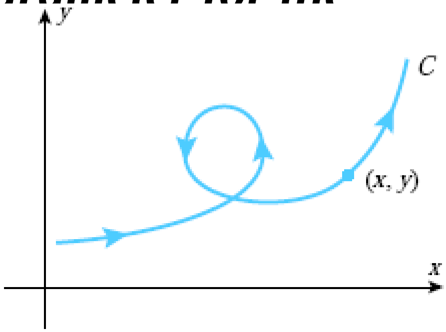
PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

PARAMETRIC EQUATIONS

Suppose that a particle moves along a curve *C in the xy-plane in such a way that its x- and y-coordinates, as functions of time, are*

$$x = f(t), \quad y = g(t)$$

We call these the *parametric equations of motion for the particle and refer to C as the trajectory of the particle or the graph of* called the *parameter for the equations.*



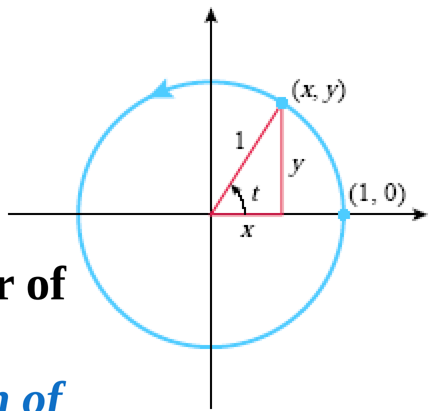
A moving particle with trajectory C

Example: Find the graph of the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

Solution. One way to find the graph is to eliminate the parameter t by noting that

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$



$x = \cos t,\ y = \sin t$
$(0 \le t \le 2\pi)$

ORIENTATION

The direction in which the graph of a pair of parametric equations is traced as the parameter increases is called *the direction of increasing parameter.*