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Limits

If the values of *f(x) can be made as close as* we like to *L by taking values of x sufficiently close to a (but not equal to a), then we* write

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of *f(x) as x approaches a is L" or "f(x) approaches L as* x approaches *a."*

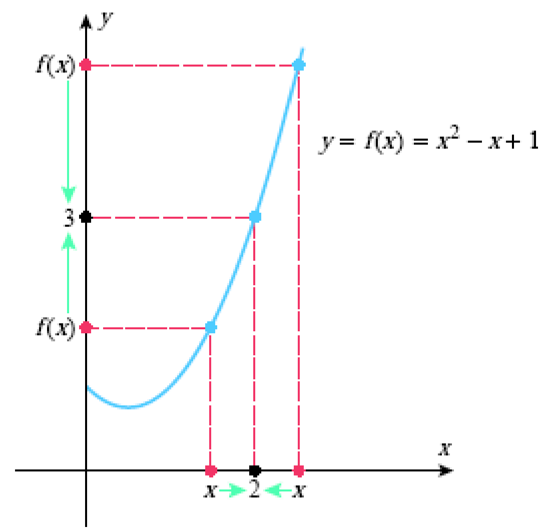
The expression can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a$$

Example:

$$f(x) = x^2 - x + 1$$

$$\lim_{x \to 2} (x^2 - x + 1) = 3$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

One-sided limits

If the values of *f(x) can be made* as close as we like to *L by taking values of x sufficiently close to a (but greater than a)*, then we write

$$\lim_{x \to a^+} f(x) = L \qquad f(x) \to L \text{ as } x \to a^+$$

f(x) approaches L as x approaches a from the right

and if the values of *f(x) can be made as close as we like to L by taking values of x* sufficiently close to *a (but less than a), then we write*

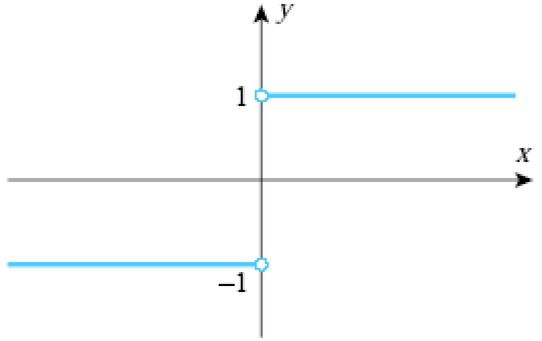
$$\lim_{x \to a^-} f(x) = L \qquad f(x) \to L \text{ as } x \to a^-$$

f(x) approaches L as x approaches a from the left.

Example:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$\lim_{x \to 0+} \frac{|x|}{x} = 1 \qquad \lim_{x \to 0-} \frac{|x|}{x} = -1$$



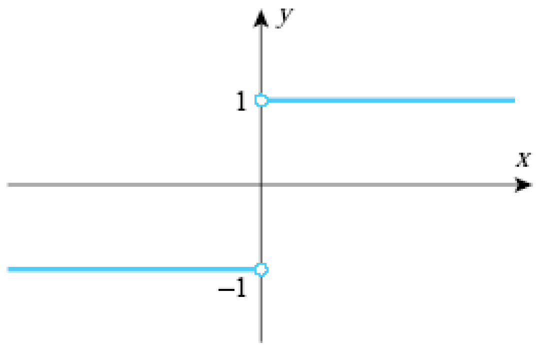
The relationship between one-sided and two-sided limits

The two sided limit of a function *f(x) exists at a if and only if both of the one-sided limits exist* at *a and have the same value; that is,*

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

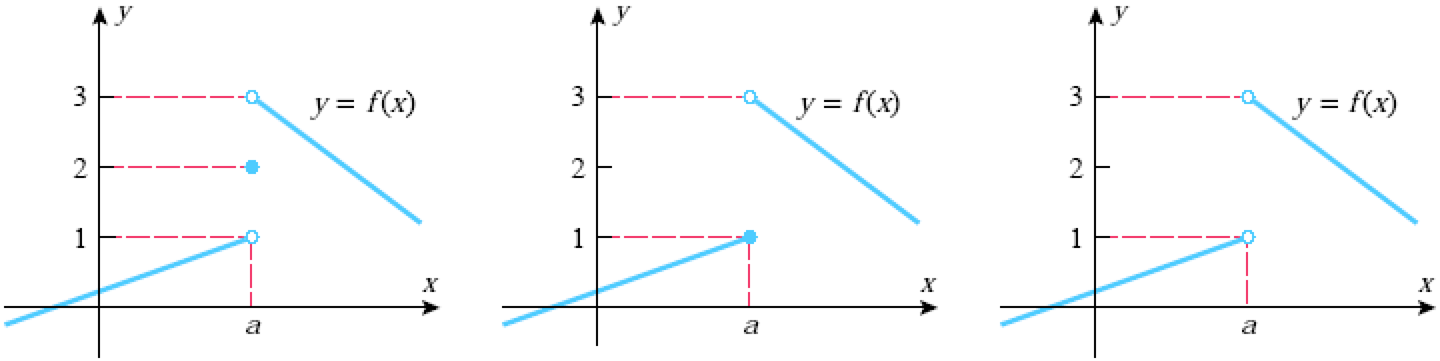


$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

Thus, the one-sided limits at 0 are not the same. Limit does not exists.

Example :

For the functions in Figure, find the one-sided and two-sided limits at *x = a if they exist.*



$$\lim_{x \to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 1$$

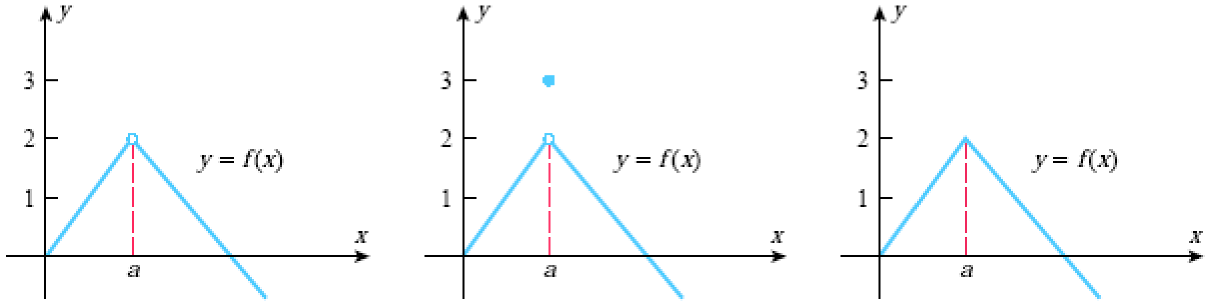
Solution. The functions in all three figures have the same one-sided limits as x→a, **since the functions are identical, except at *x = a. These limits are***

$$\lim_{x \to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as *x→a because the one-sided limits* are not equal.

Example :

For the functions in Figure, find the one-sided and two-sided limits at $x = a$ *if they exist.*



Solution. $\lim_{x \to a^+} f(x) = 2$ and $\lim_{x \to a^-} f(x) = 2$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x \to a} f(x) = 2$$

Infinite limits

The expressions $\lim_{x \to a^-} f(x) = +\infty$ and $\lim_{x \to a^+} f(x) = +\infty$

denote that *f(x) increases without bound as* x *approaches a from the left and from the* right, respectively. If both are true, then we write $\lim_{x \to a} f(x) = +\infty$

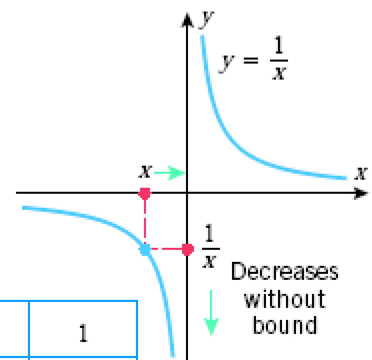
Similarly, the expressions

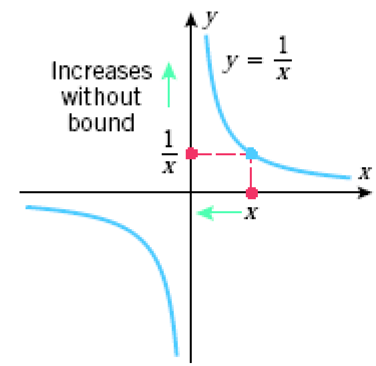
$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that *f(x) decreases without bound as* x *approaches a from the left and from the* right, respectively. If both are true, then we write $\lim_{x \to a} f(x) = -\infty$

Example :

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$



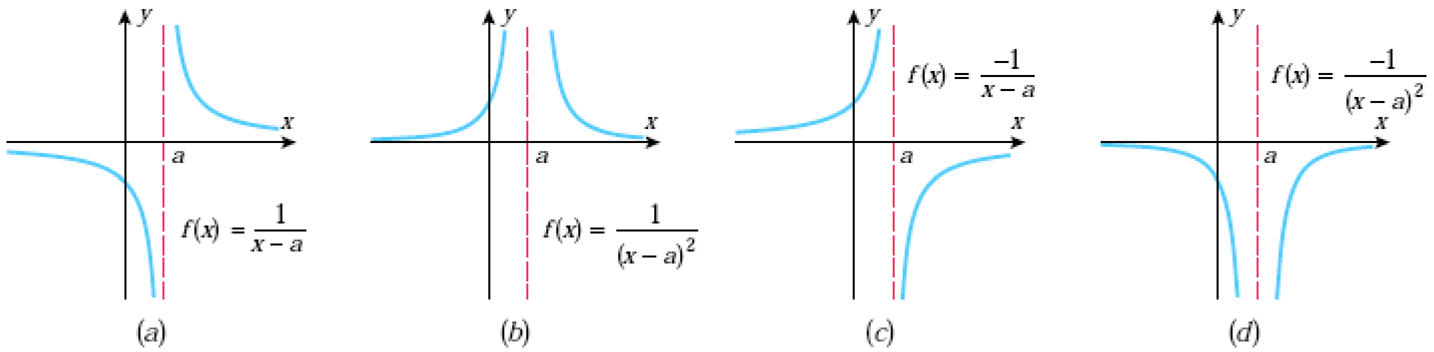


x	-1	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	-1	-10	-100	-1000	$-10,000$		10,000	1000	100	10	1

Left side → ← Right side

Example :

For the functions in Figure, describe the limits at *x = a in appropriate* limit notation.



Solution.

(a) $$\lim_{x\to a^+}\frac{1}{x-a}=+\infty \quad \text{and} \quad \lim_{x\to a^-}\frac{1}{x-a}=-\infty$$

(b) $$\lim_{x\to a}\frac{1}{(x-a)^2}=\lim_{x\to a^+}\frac{1}{(x-a)^2}=\lim_{x\to a^-}\frac{1}{(x-a)^2}=+\infty$$

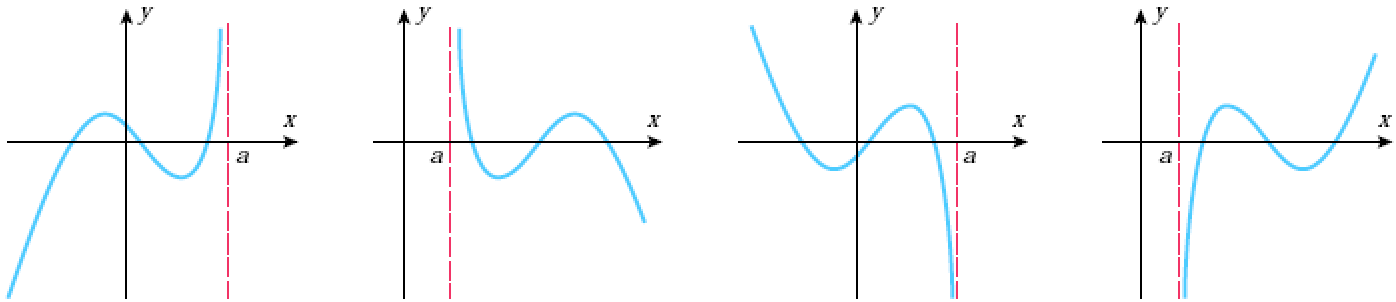
(c) $$\lim_{x\to a^+}\frac{-1}{x-a}=-\infty \quad \text{and} \quad \lim_{x\to a^-}\frac{-1}{x-a}=+\infty$$

(d) $$\lim_{x\to a}\frac{-1}{(x-a)^2}=\lim_{x\to a^+}\frac{-1}{(x-a)^2}=\lim_{x\to a^-}\frac{-1}{(x-a)^2}=-\infty$$

VERTICAL ASYMPTOTES

The graph of *y = f(x) either rises or falls without bound, squeezing closer* and closer to the vertical line *x = a as x approaches a from the side indicated in the limit.* The line *x = a is called a vertical asymptote of the curve y = f(x)*.

$$\lim_{x \to a^-} f(x) = +\infty, \quad \lim_{x \to a^+} f(x) = +\infty, \quad \lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = -\infty$$

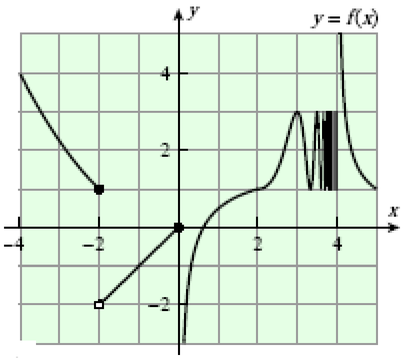


$\lim_{x \to a^-} f(x) = +\infty$ $\lim_{x \to a^+} f(x) = +\infty$ $\lim_{x \to a^-} f(x) = -\infty$ $\lim_{x \to a^+} f(x) = -\infty$

Example : For the function *f graphed in Figure, find*

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$ (d) $\lim_{x \to 0^+} f(x)$

(e) $\lim_{x \to 4^-} f(x)$ (f) $\lim_{x \to 4^+} f(x)$ (g) the vertical asymptotes of the graph of f.



Solution:

(a) &(b) $\lim_{x \to -2^-} f(x) = 1 = f(-2)$ and $\lim_{x \to -2^+} f(x) = -2$

(c) &(d) $\lim_{x \to 0^-} f(x) = 0 = f(0)$ and $\lim_{x \to 0+} f(x) = -\infty$

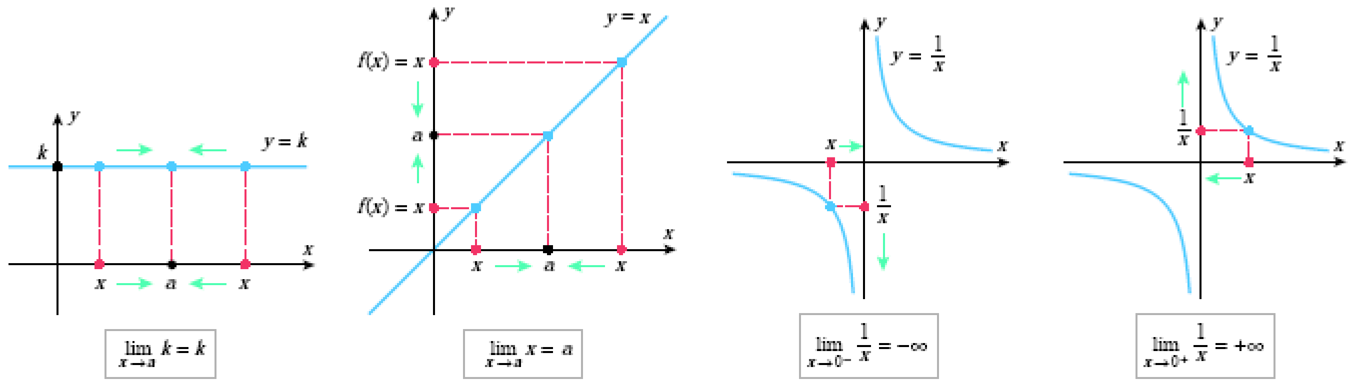
(e) &(g) $\lim_{x \to 4^-} f(x)$ does not exist due to oscillation and $\lim_{x \to 4^+} f(x) = +\infty$

(g) The y-axis and the line $x = 4$ are vertical asymptotes for the graph of f.

COMPUTING LIMITS

Theorem: ***Let a and k be real numbers***

$$(a)\ \lim_{x \to a} k = k \qquad (b)\ \lim_{x \to a} x = a \qquad (c)\ \lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad (d)\ \lim_{x \to 0^+} \frac{1}{x} = +\infty$$



Example : If *f(x) = k is a constant function, then the values of f(x) remain fixed* at *k as x varies, which explains why f(x)→k as x→a for all values of a. For example,*

$$\lim_{x \to -25} 3 = 3, \qquad \lim_{x \to 0} 3 = 3, \qquad \lim_{x \to \pi} 3 = 3$$

Example : If *f(x) = x, then as x→a it must also be true that f(x)→a. For example,*

$$\lim_{x\to 0} x = 0, \qquad \lim_{x\to -2} x = -2, \qquad \lim_{x\to \pi} x = \pi$$

Theorem: ***Let a be a real number, and suppose that***

That is, the limits exist and have values L_1 and L_2, respectively. Then:

(a) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(b) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$

(c) $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$

(d) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$, *provided* $L_2 \neq 0$

(e) $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$, *provided* $L_1 > 0$ *if* n *is even.*

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ *or as* $x \to a^+$.

Example :

$$\lim_{x \to a}[f(x) - g(x) + 2h(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) + 2 \lim_{x \to a} h(x)$$

$$\lim_{x \to a}[f(x)g(x)h(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)\left(\lim_{x \to a} h(x)\right)$$

$$\lim_{x \to a}[f(x)]^3 = \left(\lim_{x \to a} f(x)\right)^3$$

$$\lim_{x \to a}[f(x)]^n = \left(\lim_{x \to a} f(x)\right)^n$$

$$\lim_{x \to a} x^n = \left(\lim_{x \to a} x\right)^n = a^n$$

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS AS *x*→*a*

Example : Find $\lim_{x \to 5}(x^2 - 4x + 3)$.

Solution.

$$\lim_{x \to 5}(x^2 - 4x + 3) = \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3$$

$$= \lim_{x \to 5} x^2 - 4 \lim_{x \to 5} x + \lim_{x \to 5} 3 \quad = 5^2 - 4(5) + 3 \quad = 8$$

Theorem: ***For any polynomial***

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

Example : Find $\lim_{x \to 1} (x^7 - 2x^5 + 1)^{35}$.

Solution. $\lim_{x \to 1} (x^7 - 2x^5 + 1)^{35} = 0$

Example : Find $\lim_{x \to 2} \frac{5x^3 + 4}{x - 3}$.

Solution.

$$\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} = \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)}$$

$$= \frac{5 \cdot 2^3 + 4}{2 - 3} = -44$$